THE CHU CONSTRUCTION IN QUANTUM LOGIC

BORIS ISCHI AND GAVIN J. SEAL

Abstract. The Chu construction is used to define a $*-$autonomous structure on a category of complete atomistic coatomistic lattices, denoted by $\mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$. We proceed as follows. In the category $\mathbf{Set}_0$ of pointed sets, we consider the *smash product* which makes $\mathbf{Set}_0$ autonomous, hence the category $\mathbf{Chu}(\mathbf{Set}_0, 2_0)$ $*-$autonomous. Then we define a functor $\mathcal{F} : \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}} \to \mathbf{Chu}(\mathbf{Set}_0, 2_0)$ by $\mathcal{F}(\mathcal{L}) := (\Sigma \bigcup \{0\}, r, \Sigma' \bigcup \{1\})$ where Σ and Σ' denote the sets of atoms and coatoms of $\mathcal{L}$ respectively. Finally, we prove that $\mathcal{F}$ is full and faithful, and that $\mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$ is closed under dual and tensor in $\mathbf{Chu}(\mathbf{Set}_0, 2_0)$, therefore $*-$autonomous. This construction leads to a new tensor product $\circledast$, which we compare with a certain number of other tensor products. For DAC-lattices, we describe $\mathcal{L}_1 \circledast \mathcal{L}_2$ and $\mathcal{L}_1 \multimap \mathcal{L}_2 = (\mathcal{L}_1 \circledast \mathcal{L}_2^{\mathrm{op}})^{\mathrm{op}}$ in terms of semilinear maps.

1. Introduction

In the Piron-Aerts approach to the foundations of quantum mechanics, a physical system is described by a complete atomistic lattice, the atoms of which represent the possible physical states [15],[1]. Moreover, the time-evolution is modelled by a join-preserving map sending atoms to atoms [9]. Complete atomistic lattices and join-preserving maps sending atoms to atoms trivially form a category. It is therefore natural to study and interpret from the physical point of view the constructions arising from powerful category-theoretic techniques applied to this particular category. The categorical approach to quantum logic has become very popular in the last decade [6]. However, the Chu construction applied to the category mentioned above has, to our knowledge, not been studied yet. As our main result, we find that the tensor product of complete atomistic lattices arising from the Chu construction yields a possible model for the property lattice of *separated* quantum systems in the sense of Aerts [1].

Barr's $*-$autonomous categories form a model for a large fragment of linear logic, and play an important role in theoretical computer science. Given a finitely complete autonomous category $\mathbf{C}$ and an object A of $\mathbf{C}$, the Chu construction yields a category $\mathbf{Chu} \equiv \mathbf{Chu}(\mathbf{C}, \mathsf{A})$, a bifunctor $\otimes$, a functor $^{\perp} : \mathbf{Chu}^{\mathrm{op}} \to \mathbf{Chu}$, and two objects $\top$ and $\perp$, such that $\langle \mathbf{Chu}(\mathbf{C}, \mathsf{A}), \otimes, \top, \multimap, \perp \rangle$ is $*-$autonomous, where $\mathsf{A} \multimap \mathsf{B} := (\mathsf{A} \otimes \mathsf{B}^{\perp})^{\perp}$.

For instance, if $\mathbf{C} = \mathbf{Set}$ and $\mathsf{A} = 2 \equiv \{0, 1\}$, the objects of $\mathbf{Chu_2} \equiv \mathbf{Chu}(\mathbf{Set}, 2)$ are triples (A, r, X), where A and X are sets, and r is a map; $r : A \times X \to 2$. Arrows are pair of maps $(f, g) : (A, r, X) \to (B, s, Y)$ with $f : A \to B$ and $g : Y \to X$

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such that $s(f(a),y) = r(a,g(y))$ for all $a \in A$ and $y \in Y$. The functor $^\perp$ is defined on objects as $(A,r,X)^\perp = (X,\check{r},A)$, with $\check{r}(x,a) = r(a,x)$, and the bifunctor $\otimes$ as $\mathsf{A}_1 \otimes \mathsf{A}_2 = (A_1 \times A_2, t, \mathbf{Chu}_2(\mathsf{A}_1, \mathsf{A}_2^\perp))$ where $\mathsf{A}_i = (A_i, r_i, X_i)$ and $t((a_1,a_2),(f,g)) = r_2(a_2, f(a_1))$.

Obviously, if $\mathcal{L}$ is an atomistic lattice, then $\mathcal{G}(\mathcal{L}) := (\Sigma, r, \mathcal{L})$, where Σ denotes the set of atoms of $\mathcal{L}$ and $r(p,a) = 1 \Leftrightarrow p \leq a$, is an object of $\mathbf{Chu}_2$. On the other hand, if in addition of being atomistic lattices, $\mathcal{L}_1$ and $\mathcal{L}_2$ are moreover complete, then $(f,g) \in \mathbf{Chu}_2(\mathcal{G}(\mathcal{L}_1), \mathcal{G}(\mathcal{L}_2))$ if and only if there is a Galois connection (h, h°) between $\mathcal{L}_1$ and $\mathcal{L}_2$ with h sending atoms to atoms, $f = h|_{\Sigma_1}$ and $g = h^\circ$. As a consequence, for the category $\mathbf{Cal}$ of complete atomistic lattices with maps preserving arbitrary joins and sending atoms to atoms, we have a full and faithful functor $\mathcal{G} : \mathbf{Cal} \to \mathbf{Chu}_2$. Moreover, it can be proved that $\mathbf{Cal}$ is closed under the tensor of $\mathbf{Chu}_2$, *i.e.* $\mathcal{G}(\mathcal{L}_1 \owedge \mathcal{L}_2) \cong \mathcal{G}(\mathcal{L}_1) \otimes \mathcal{G}(\mathcal{L}_2)$ where $\owedge$ denotes the (complete) semilattice tensor product of Fraser [10] (or the tensor product of Shmuely [16], see Proposition 6.6, Remark 6.7 and 7.4 below). However, $\mathbf{Cal}$ is obviously not closed under $^\perp$.

A natural way to get closure under $^\perp$, is to replace in the definition of the functor $\mathcal{G}$ the set $\mathcal{L}$ by the set of coatoms of $\mathcal{L}$. Indeed, for the (non full) subcategory $\mathbf{Cal}_{\mathbf{Sym}}$ of coatomistic lattices with arrows f having a right adjoint f° sending coatoms to coatoms, we can define a full and faithful functor $\mathcal{K} : \mathbf{Cal}_{\mathbf{Sym}} \to \mathbf{Chu}_2$ as $\mathcal{K}(\mathcal{L}) = (\Sigma, r, \Sigma')$ and $\mathcal{K}(f) = (f, f^\circ)$, where Σ' stands for the set of coatoms of $\mathcal{L}$. Then $\mathcal{K}(\mathcal{L}^{\mathrm{op}}) = \mathcal{K}(\mathcal{L})^\perp$ (where $\mathcal{L}^{\mathrm{op}}$ denotes the dual of $\mathcal{L}$ defined by the converse order-relation) but the category $\mathbf{Cal}_{\mathbf{Sym}}$ is not closed under the tensor of $\mathbf{Chu}_2$. Indeed, denote by MO_n the complete atomistic lattice with n atoms such that 1 covers each atom. Let $\mathcal{L}_1 = \mathsf{MO}_n$ and $\mathcal{L}_2 = \mathsf{MO}_m$ with $n \neq m$. Then, it is easy to check that $\mathbf{Chu}_2(\mathcal{K}(\mathcal{L}_1), \mathcal{K}(\mathcal{L}_2)^\perp) = \emptyset$ since there is no bijection between Σ_1 and Σ_2. As a consequence, there is obviously no $\mathcal{L} \in \mathbf{Cal}_{\mathbf{Sym}}$ such that $\mathcal{K}(\mathcal{L}) \cong \mathcal{K}(\mathcal{L}_1) \otimes \mathcal{K}(\mathcal{L}_2)$.

In the preceding example, the reason why $\mathbf{Cal}_{\mathbf{Sym}}$ is not closed under the tensor of $\mathbf{Chu}_2$ is that $\mathbf{Cal}_{\mathbf{Sym}}(\mathcal{L}_1, \mathcal{L}_2^{\mathrm{op}}) = \emptyset$. A simple way to remedy to this is to consider more morphisms, namely maps preserving arbitrary joins and sending atoms to atoms or 0. Call $\mathbf{Cal}^0$ the category of complete atomistic lattices equipped with those morphisms. Then $\mathbf{Cal}^0$ can be embedded canonically in the category $\mathbf{Chu}_{2_0} \equiv \mathbf{Chu}(\mathbf{Set}_0, 2_0)$, where $\mathbf{Set}_0$ denotes the category of pointed sets with monoidal structure given by the smash product, and 2_0 denotes the set $\{0,1\}$ pointed by 0. Indeed, it is easy to check that the functor $\mathcal{G}^0 : \mathbf{Cal}^0 \to \mathbf{Chu}_{2_0}$ defined on objects as $\mathcal{G}^0(\mathcal{L}) = (\Sigma \bigcup \{0\}, r, \mathcal{L})$ where $\Sigma \bigcup \{0\}$ is pointed by 0 and $\mathcal{L}$ is pointed by 1, and where $r(x,a) = 0 \Leftrightarrow x \leq a$, and on morphisms as $\mathcal{G}^0(f) = (f, f^\circ)$, is full and faithful, and that $\mathbf{Cal}^0$ is closed under the tensor of $\mathbf{Chu}_{2_0}$.

However, as $\mathbf{Cal}$, the category $\mathbf{Cal}^0$ is not closed under the dualizing functor $^\perp$. Again, a natural way to obtain closure under $^\perp$ is to replace in the definition of $\mathcal{G}^0$ the set $\mathcal{L}$ by $\Sigma' \bigcup \{1\}$. Hence, we define a functor $\mathcal{F} : \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}} \to \mathbf{Chu}_{2_0}$ as $\mathcal{F}(\mathcal{L}) = (\Sigma \bigcup \{0\}, r, \Sigma' \bigcup \{1\})$ and $\mathcal{F}(f) = (f, f^\circ)$. Then, for $\mathcal{F}$ to be full, we have to consider more morphisms than in $\mathbf{Cal}_{\mathbf{Sym}}$, namely maps preserving arbitrary joins and sending atoms to atoms or 0 with right adjoint sending coatoms to coatoms or 1.

In order to check closure under tensor, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete atomistic coatomistic lattices. Then,

$$\mathcal{F}(\mathcal{L}_1) \otimes_C \mathcal{F}(\mathcal{L}_2) \cong (\Sigma_1 \times \Sigma_2 \bigcup \{0\}, t, \mathbf{Chu}_{2_0}(\mathcal{F}(\mathcal{L}_1), \mathcal{F}(\mathcal{L}_2)^{\perp}))\,,$$

where $\otimes_C$ denotes the tensor of $\mathbf{Chu}_{2_0}$. Now, $(f, g) \in \mathbf{Chu}_{2_0}(\mathcal{F}(\mathcal{L}_1), \mathcal{F}(\mathcal{L}_2)^{\perp})$ if and only if $f : \Sigma_1 \to \Sigma'_2 \bigcup\{1\}$, $g : \Sigma_2 \to \Sigma'_1 \bigcup\{1\}$, and $q \leq f(p) \Leftrightarrow p \leq g(q)$, for all atoms $p \in \Sigma_1$ and $q \in \Sigma_2$. To the map f we can associate a subset x^f of $\Sigma_1 \times \Sigma_2$ defined as $x^f = \bigcup\{\{p\} \times \Sigma_2[f(p)]\,;\, p \in \Sigma_1\}$, where $\Sigma_2[b]$ denotes the set of atoms under b. Hence, it can be seen that $\mathcal{F}(\mathcal{L}_1) \otimes_C \mathcal{F}(\mathcal{L}_2) \cong (\Sigma_1 \times \Sigma_2 \bigcup\{0\}, \Gamma)$, where Γ is the set of all subsets x^f of $\Sigma_1 \times \Sigma_2$. As a consequence, $\mathcal{F}(\mathcal{L}_1) \otimes_C \mathcal{F}(\mathcal{L}_2)$ is in the image of $\mathcal{F}$ only if there is $\mathcal{L} \in \mathbf{Cal^0_{Sym}}$ such that $\Sigma = \Sigma_1 \times \Sigma_2$ and $\{\Sigma[x]\,;\, x \in \Sigma' \bigcup\{1\}\} = \Gamma$, that is only if for any a, $b \in \Gamma$ different from $\Sigma_1 \times \Sigma_2$, a is not a subset of b and b is not a subset of a. This fails to be true, for instance if $\mathcal{L}_1$ and $\mathcal{L}_2$ are powerset lattices (see Example 5.7).

Therefore, in order to have closure under tensor, the objects in $\mathbf{Cal^0_{Sym}}$ cannot be all complete atomistic coatomistic lattices, but me must impose some condition. We will prove that a sufficient condition (which we call $\mathbf{A_0}$) is to ask that for any two atoms p and q and any two coatoms x and y, there is a coatom z and an atom r such that $p \bigwedge z = 0 = q \bigwedge z$ and $r \bigwedge x = 0 = r \bigwedge y$. Note that our Axiom $\mathbf{A_0}$ implies that the lattices are irreducible. We will give an example of a complete atomistic orthocomplemented lattice $\mathcal{L}$ which is irreducible but does not satisfy $\mathbf{A_0}$, and such that there is no $\mathcal{L}_0 \in \mathbf{Cal^0_{Sym}}$ with $\mathcal{F}(\mathcal{L}_0) \cong \mathcal{F}(\mathcal{L}) \otimes_C \mathcal{F}(\mathcal{L})$ (see Example 5.8).

Using the functor $\mathcal{F}$ we prove that $\mathbf{Cal^0_{Sym}}$ is closed under both $^{\perp}$ and the tensor of $\mathbf{Chu}_{2_0}$, hence that $\mathbf{Cal^0_{Sym}}$ inherits the $*-$autonomous structure of $\mathbf{Chu}_{2_0}$. This result is presented in Theorem 5.5, Section 5.

The rest of this paper is organized as follows. In Section 2, we briefly recall the definition of $*-$autonomous categories and of $\mathbf{Chu}_{2_0}$. The category $\mathbf{Cal^0_{Sym}}$ and the bifunctor $\circledast$ are introduced in Sections 3 and 4 respectively. Section 5 is devoted to our main result. In Section 6, the tensor product $\circledast$ is compared to other lattice-theoretical tensor products. It is characterized in terms of a universal property with respect to what we call *weak bimorphisms* in Section 7. Finally, we focus on DAC-lattices in the last Section 8.

2. The category $\mathbf{Chu}(\mathbf{Set}_0, 2_0)$

We begin by briefly recalling the definition of a $*-$autonomous category. For details, we refer to Barr [2], [3]. For general terminology concerning category theory, we refer to Mac Lane [13].

Definition 2.1. An autonomous category $\mathbf{C}$ is a monoidal symmetric closed category. *Monoidal symmetric* means that there is a bifunctor $-\otimes- : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, an object $\top$, and natural isomorphisms $\alpha_{\mathsf{ABC}} : (\mathsf{A} \otimes \mathsf{B}) \otimes \mathsf{C} \to \mathsf{A} \otimes (\mathsf{B} \otimes \mathsf{C})$, $r_{\mathsf{A}} : \mathsf{A} \otimes \top \to \mathsf{A}$, $l_{\mathsf{A}} : \top \otimes \mathsf{A} \to \mathsf{A}$, and $s_{\mathsf{AB}} : \mathsf{A} \otimes \mathsf{B} \to \mathsf{B} \otimes \mathsf{A}$, satisfying some coherence conditions (see the appendix). *Closed* means that there is a bifunctor $- \multimap - : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{C}$ such that for all objects A, B, C of $\mathbf{C}$, there is an isomorphism $\mathbf{C}(\mathsf{A} \otimes \mathsf{B}, \mathsf{C}) \cong \mathbf{C}(\mathsf{B}, \mathsf{A} \multimap \mathsf{C})$, natural in B and C.

Remark 2.2. Let $\mathbf{C}$ be an autonomous category and $\perp$ an object of $\mathbf{C}$. Since $\mathbf{C}$ is closed and symmetric, for each object A we have

$$\begin{aligned}\mathbf{C}(\mathsf{A} \multimap \perp, \mathsf{A} \multimap \perp) &\cong \mathbf{C}(\mathsf{A} \otimes (\mathsf{A} \multimap \perp), \perp) \\ &\cong \mathbf{C}((\mathsf{A} \multimap \perp) \otimes \mathsf{A}, \perp) \cong \mathbf{C}(\mathsf{A}, (\mathsf{A} \multimap \perp) \multimap \perp)\,.\end{aligned}$$

Hence, to the identity arrow $(\mathsf{A} \multimap \perp) \to (\mathsf{A} \multimap \perp)$ corresponds an arrow $\mathsf{A} \to ((\mathsf{A} \multimap \perp) \multimap \perp)$.

Definition 2.3 (Barr, [2])**.** If for every object A of $\mathbf{C}$ the aforementioned arrow $\mathsf{A} \to ((\mathsf{A} \multimap \perp) \multimap \perp)$ is an isomorphism, the object $\perp$ is called a *dualizing object*. A $*$-autonomous category $\mathbf{C}$ is an autonomous category with a dualizing object. Usually, $\mathsf{A} \multimap \perp$ is written $\mathsf{A}^\perp$.

Chu's paper [2] (see also Barr [4], [5]) describes a construction of a $*-$autonomous category starting with a finitely complete autonomous category. We outline the construction of Chu for the category $\mathbf{Set}_0$ of pointed sets and pointed maps.

Definition 2.4. On the category $\mathbf{Set}_0$ of pointed sets with pointed maps, we define the *smash product* $- \sharp - : \mathbf{Set}_0 \times \mathbf{Set}_0 \to \mathbf{Set}_0$ as

$$A \sharp B := [(A \backslash \{0_A\}) \times (B \backslash \{0_B\})] \bigcup \{0_\sharp\}\,,$$

where 0_A and 0_B are the respective base-points of A and B. Moreover, we write 2_0 for the set $\{0,1\}$ pointed by 0.

Lemma 2.5. *The category $\langle \mathbf{Set}_0, \sharp, 2_0, \multimap \rangle$, with $A \multimap B := \mathbf{Set}_0(A, B)$ pointed by the constant map, is autonomous.*

Proof. The proof is direct and is omitted here. □

Notation 2.6. Let A, B and C be pointed sets and let $r : A \sharp B \to C$ be a pointed map. We do not distinguish this map from the map defined on $A \times B$ with value in C defined by $r(a,b)$ if $a \neq 0_A$ and $b \neq 0_B$, and as 0_C if $a = 0_A$ or $b = 0_B$.

Definition 2.7. An object of $\mathbf{Chu}_{2_0} := \mathbf{Chu}(\mathbf{Set}_0, 2_0)$ is a triplet (A, r, X), where A and X are pointed sets, and $r : A \sharp X \to 2_0$ is a pointed map. An arrow is a pair of pointed maps $(f, g) : (A, r, X) \to (B, s, Y)$, with $f : A \to B$ and $g : Y \to X$, satisfying $s(f(a), y) = r(a, g(y))$ for all $a \in A$ and $y \in Y$.

Remark 2.8. Let $\mathsf{A} = (A, r, X)$ and $\mathsf{B} = (B, s, Y)$ be objects of $\mathbf{Chu}_{2_0}$. The pair of constant maps $f : A \to B$; $a \mapsto 0_B$ and $g : Y \to X$; $y \mapsto 0_X$ forms an arrow of $\mathbf{Chu}_{2_0}(\mathsf{A}, \mathsf{B})$ which we call the *constant* arrow.

Definition 2.9. For $i = 1, 2$, let $\mathsf{A}_i = (A_i, r_i, X_i)$ and $\mathsf{B}_i = (B_i, s_i, Y_i)$ be objects of $\mathbf{Chu}_{2_0}$, and $(f_i, g_i) \in \mathbf{Chu}_{2_0}(\mathsf{A}_i, \mathsf{B}_i)$.

The functor ${}^\perp : \mathbf{Chu}_{2_0}^{\mathrm{op}} \to \mathbf{Chu}_{2_0}$ is defined on objects as $\mathsf{A}_1^\perp := (X_1, \check{r}_1, A_1)$, where $\check{r}_1(x, a) = r_1(a, x)$, and on arrows as $(f_1, g_1)^\perp := (g_1, f_1) : \mathsf{B}_1^\perp \to \mathsf{A}_1^\perp$.

The bifunctor $- \otimes_C - : \mathbf{Chu}_{2_0} \times \mathbf{Chu}_{2_0} \to \mathbf{Chu}_{2_0}$ is defined on objects as

$$\mathsf{A}_1 \otimes_C \mathsf{A}_2 = (A_1 \sharp A_2, t, \mathbf{Chu}_{2_0}(\mathsf{A}_1, \mathsf{A}_2^\perp))\,,$$

with $\mathbf{Chu}_{2_0}(\mathsf{A}_1, \mathsf{A}_2^\perp)$ pointed by the constant arrow, and with t defined as

$$t((a_1, a_2), (f, g)) := r_1(a_1, g(a_2)) = \check{r}_2(f(a_1), a_2)\,.$$

Further, $f_1 \otimes_C f_2 : \mathsf{A}_1 \otimes_C \mathsf{A}_2 \to \mathsf{B}_1 \otimes_C \mathsf{B}_2$ is defined as $(a_1, a_2) \mapsto (f_1(a_1), f_2(a_2))$ if $f_1(a_1) \neq 0$ and $f_2(a_2) \neq 0$ (with $(f_1 \otimes_C f_2)(a_1, a_2) := 0_\#$ if $f_1(a_1) = 0$ or $f_2(a_2) = 0$), and as $(f, g) \mapsto (g_2 \circ f \circ f_1, g_1 \circ g \circ f_2)$ for $(f, g) \in \mathbf{Chu}_{2_0}(\mathsf{B}_1, \mathsf{B}_2^\perp)$.

Definition 2.10. The object $\top$ is defined as $\top := (2_0, r, 2_0)$ with r injective. Moreover, the dualizing object $\perp$ is defined as $\top^\perp$. Finally, the bifunctor $\multimap$ is given by $\mathsf{A} \multimap \mathsf{B} := (\mathsf{A} \otimes_C \mathsf{B}^\perp)^\perp$.

Remark 2.11. The object $\top$ is the tensor unit; $\mathsf{A} \otimes_C \top \cong \mathsf{A}$. Hence, we have $\mathsf{A} \multimap \top^\perp \cong \mathsf{A}^\perp$, and $\perp$ is the dualizing object.

Since $\mathbf{Set}_0$ is finitely complete and autonomous (Lemma 2.5), we have the following result.

Proposition 2.12. *The category* $\langle \mathbf{Chu}_{2_0}, \otimes_C, \top, \multimap, \perp \rangle$ *is* $*$*-autonomous.*

3. The category $\mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$

For lattice-theoretic terminology, we refer to Maeda and Maeda [14].

Notation 3.1. Let Σ be a nonempty set and $\mathcal{L} \subseteq 2^\Sigma$. We say that $\mathcal{L}$ is a *simple closure space* on Σ if $\mathcal{L}$ contains $\emptyset$, Σ, and all singletons, and if $\mathcal{L}$ is closed under arbitrary set-intersections (*i.e.* $\bigcap \omega \in \mathcal{L}$ for all $\omega \subseteq \mathcal{L}$). Note that a simple closure space (ordered by set-inclusion) is a complete atomistic lattice. For $p \in \Sigma$, we identify p with $\{p\} \in \mathcal{L}$.

Let $\mathcal{L}_i$ be a poset and $a \in \mathcal{L}_i$. The bottom and top elements of $\mathcal{L}_i$, if they exist, are denoted by 0 and 1 respectively. We denote by $\mathcal{L}_i^{\mathrm{op}}$ the dual of $\mathcal{L}_i$ (defined by the converse order relation), by Σ_i and Σ_i' the sets of atoms and coatoms of $\mathcal{L}_i$ respectively, by $\Sigma[a]$ the set of atoms under a, and by $\Sigma'[a]$ the set of coatoms above a. We write

$$\mathsf{Cl}(\mathcal{L}_i) = \{\Sigma[a]\,;\, a \in \mathcal{L}_i\}\,,$$

ordered by set-inclusion. For any subset $\omega \subseteq \mathcal{L}$, we define $\mathsf{Cl}(\omega)$ in an obvious similar way. Note that if $\mathcal{L}$ is a complete atomistic lattice, then $\mathsf{Cl}(\mathcal{L})$ is a simple closure space on the set of atoms of $\mathcal{L}$.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be posets. A Galois connection between $\mathcal{L}_1$ and $\mathcal{L}_2$ (or equivalently an adjunction) is a pair (f, g) of order-preserving maps with $f : \mathcal{L}_1 \to \mathcal{L}_2$ and $g : \mathcal{L}_2 \to \mathcal{L}_1$ such that for any $a \in \mathcal{L}_1$ and $b \in \mathcal{L}_2$, $f(a) \le b \Leftrightarrow a \le g(b)$.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete lattices and $f : \mathcal{L}_1 \to \mathcal{L}_2$ a map. The join and meet in $\mathcal{L}_i^{\mathrm{op}}$ are denoted by $\bigvee^{\mathrm{op}}$ and $\bigwedge^{\mathrm{op}}$ respectively. The map $f^\circ : \mathcal{L}_2 \to \mathcal{L}_1$ is defined as

$$f^\circ(b) := \bigvee\{a \in \mathcal{L}_1\,;\, f(a) \le b\}\,,$$

and $f^{\mathrm{op}} : \mathcal{L}_2^{\mathrm{op}} \to \mathcal{L}_1^{\mathrm{op}}$ as $f^{\mathrm{op}}(b) := f^\circ(b)$. Finally, 2 stands for the lattice with only two elements.

Lemma 3.2. *Let* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *be posets and* (f, g) *a Galois connection between* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*. Then*

(i) $g = f^\circ$.
(ii) f *and* f^{op} *preserve all existing joins and* g *preserves all existing meets.*

Proof. (i) Let $b \in \mathcal{L}_2$. Define $\Omega_b := \{a \in \mathcal{L}_1;\ f(a) \le b\}$. Then $a \le g(b)$, for any $a \in \Omega_b$. Moreover, $f(g(b)) \le b$. As a consequence, $g(b) = \bigvee \Omega_b$.

(ii) Let $\omega \subseteq \mathcal{L}_1$ such that $\bigvee \omega$ exists. Since f is order-preserving, $f(a) \le f(\bigvee \omega)$, for all $a \in \omega$. Let $x \in \mathcal{L}_2$ such that $f(a) \le x$ for all $a \in \omega$. Then $a \le g(x)$, for all $a \in \omega$. Therefore, $\bigvee \omega \le g(x)$, hence $f(\bigvee \omega) \le x$. As a consequence, $f(\bigvee \omega) = \bigvee\{f(a);\ a \in \omega\}$.

The statements for g and f^{op} follow by duality. $\square$

Lemma 3.3. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete lattices and let $f : \mathcal{L}_1 \to \mathcal{L}_2$. Then*

(i) *f preserves arbitrary joins $\Leftrightarrow$ (f, f°) is a Galois connection between $\mathcal{L}_1$ and $\mathcal{L}_2$.*

Suppose moreover that $\mathcal{L}_1$ and $\mathcal{L}_2$ are atomistic and that f sends atoms to atoms or 0. Denote by F the restriction of f to atoms. Then

(ii) *f preserves arbitrary joins $\Leftrightarrow$ $f(a) = \bigvee F(\Sigma[a])$ and $F^{-1}(\Sigma[b] \bigcup \{0\}) \in \mathsf{Cl}(\mathcal{L}_1)$, $\forall a \in \mathcal{L}_1$, $b \in \mathcal{L}_2$.*

Proof. (i) Suppose that f preserves arbitrary joins. Let $a \in \mathcal{L}_1$ and $b \in \mathcal{L}_2$. By definition of f°, if $f(a) \le b$, then $a \le f^\circ(b)$. On the other hand, if $a \le f^\circ(b)$, then $f(a) \le f(f^\circ(b)) \le b$ since f preserves arbitrary joins.

(ii) Suppose that f preserves arbitrary joins. Let p be an atom under $\bigvee F^{-1}(\Sigma[b] \bigcup\{0\})$. Then,

$$f(p) \le \bigvee F(F^{-1}(\Sigma[b] \bigcup \{0\})) \le b\,,$$

hence $p \in F^{-1}(\Sigma[b] \bigcup\{0\})$.

We now prove the converse. Define $g(b) = \bigvee F^{-1}(\Sigma[b] \bigcup\{0\})$. We prove that the pair (f, g) forms a Galois connection between $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $a \in \mathcal{L}_1$ and $b \in \mathcal{L}_2$. Suppose that $f(a) \le b$. Then, $\bigvee F(\Sigma[a]) \le b$, hence, $f(p) \le b$ for all $p \in \Sigma[a]$. Therefore $a \le g(b)$. Suppose now that $a \le g(b)$. Then, for any $p \in \Sigma[a]$, $p \le g(b)$, hence, from the second hypothesis, $f(p) \le b$. As a consequence, $f(a) = \bigvee F(\Sigma[a]) \le b$. $\square$

Definition 3.4. We denote by $\mathbf{Cal^0_{Sym}}$ the following category: the objects are all complete atomistic coatomistic lattices $\mathcal{L}$ such that

$$(\mathbf{A_0}) \qquad \Sigma[x] \bigcup \Sigma[y] \ne \Sigma, \quad \forall\, x,\, y \in \Sigma'\,,$$

and such that Axiom $\mathbf{A}_0$ holds also in $\mathcal{L}^{\mathrm{op}}$ (*i.e.* $\Sigma'[p] \bigcup \Sigma'[q] \ne \Sigma'$, $\forall\, p,\, q \in \Sigma$); the arrows are all maps f preserving arbitrary joins and sending atoms to atoms or 0 such that f^{op} sends atoms to atoms or 0 (*i.e.* f° sends coatoms to coatoms or 1).

Remark 3.5. Note that $2 \in \mathbf{Cal^0_{Sym}}$. Moreover, the map $-^{\mathrm{op}} : {\mathbf{Cal^0_{Sym}}}^{\mathrm{op}} \to \mathbf{Cal^0_{Sym}}$ is a functor. Indeed, consider two arrows of $\mathbf{Cal^0_{Sym}}$, say $g : \mathcal{L}_1 \to \mathcal{L}_2$ and $f : \mathcal{L}_2 \to \mathcal{L}_3$. Let $c \in \mathcal{L}_3^{\mathrm{op}}$. Then we have

$$\begin{aligned} g^{\mathrm{op}} \circ f^{\mathrm{op}}(c) = g^\circ(f^\circ(c)) &= \bigvee\{a \in \mathcal{L}_1\,;\, g(a) \le f^\circ(c)\} \\ &= \bigvee\{a \in \mathcal{L}_1\,;\, f(g(a)) \le c\} = (f \circ g)^\circ(c) = (f \circ g)^{\mathrm{op}}(c)\,. \end{aligned}$$

Finally, note that Axiom $\mathbf{A}_0$ will only be needed for Lemma 4.7; note also that it implies that $\mathcal{L}$ is irreducible (see [14], Theorem 4.13).

Definition 3.6. A lattice $\mathcal{L}$ with 0 and 1 such that $\mathcal{L}$ and $\mathcal{L}^{\mathrm{op}}$ are atomistic with the covering property, is called a *DAC-lattice*.

Example 3.7. Let $\mathcal{L}$ be an irreducible complete DAC-lattice. Then $\mathcal{L}$ is an object of $\mathbf{Cal^0_{Sym}}$.

Indeed, let x, y be coatoms. Suppose that $\Sigma[x] \bigcup \Sigma[y] = \Sigma$. Let z be a coatom above $x \bigwedge y$. Let p be an atom under z such that $p \bigwedge x \bigwedge y = 0$. Then, since by hypothesis $\Sigma[x] \bigcup \Sigma[y] = \Sigma$, $p \le x$ or $p \le y$. Note that since $\mathcal{L}^{\mathrm{op}}$ has the covering property, z covers $x \bigwedge y$. Hence, if $p \le x$, then $z = p \bigvee (x \bigwedge y) = x$, and if $p \le y$, then $z = p \bigvee (x \bigwedge y) = y$. As a consequence, the set of coatoms above $x \bigwedge y$ is

given by $\{x, y\}$, a contradiction. Indeed, recall that the join of any two atoms of an irreducible complete DAC-lattice contains a third atom (see [14], Theorems 28.8 and 27.6, and Lemma 11.6), hence, by duality, for any two coatoms x and y, there is a third coatom above their meet.

Finally, by duality, $\mathcal{L}^{\mathrm{op}}$ also satisfies $\mathbf{A}_0$.

We end this section by recalling the relation between irreducible complete DAC-lattices and lattices of closed subspaces of vector spaces.

Definition 3.8 (see [14], Definition 33.1)**.** Let E be a left vector space (respectively F a right vector space) over a division ring $\mathbb{K}$. Then (E, F) is called a *pair of dual spaces* if there exists a non-degenerate bilinear map $f : E \times F \to \mathbb{K}$. For $A \subseteq E$, define

$$A^{\perp} := \{y \in F\, ;\, f(x, y) = 0,\, \forall x \in A\}\,,$$

and for $B \subseteq F$ define $B^{\perp}$ similarly. Define

$$\mathcal{L}_F(E) := \{A \subseteq E;\, A^{\perp\perp} = A\}\,,$$

ordered by set-inclusion.

Theorem 3.9 (see [14], Theorem 33.4)**.** *Let (E, F) be a pair of dual spaces. Then $\mathcal{L}_F(E)$ is an irreducible complete DAC-lattice.*

Theorem 3.10 (see [14], Theorem 33.7)**.** *If $\mathcal{L}$ is an irreducible complete DAC-lattice of length ≥ 4, then there exists a pair of dual spaces (E, F) over a division ring $\mathbb{K}$ such that $\mathcal{L} \cong \mathcal{L}_F(E)$.*

Remark 3.11. If x is a finite dimensional subspace of E, then $x \in \mathcal{L}_F(E)$ (see [14], Lemma 33.3.2).

4. The bifunctor $\circledast$

The bifunctor $\circledast$ will provide $\mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$ with a suitable "tensor product" in order to make it into a $*-$autonomous category.

Notation 4.1. For $p \in \Sigma_1 \times \Sigma_2$, we denote the first component of p by p_1 and the second by p_2. For $R \subseteq \Sigma_1 \times \Sigma_2$, we adopt the following notations.

$$R_1[p] := \{q_1 \in \Sigma_1\, ;\, (q_1, p_2) \in R\}\,,$$
$$R_2[p] := \{q_2 \in \Sigma_2\, ;\, (p_1, q_2) \in R\}\,.$$

Remark 4.2. Note that $R_1[p]$ (respectively $R_2[p]$) depends only on p_2 (respectively only on p_1). For $(r, s) \in \Sigma_1 \times \Sigma_2$, we define $R_1[s]$ as $R_1[(p, s)]$ and $R_2[r]$ as $R_2[(r, q)]$ for any $(p, q) \in \Sigma_1 \times \Sigma_2$.

Definition 4.3. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete atomistic coatomistic lattices. Then we define

$$\Sigma'_{\circledast} := \Big\{R \subsetneqq \Sigma_1 \times \Sigma_2\, ;\;\; R_1[p] \in \mathsf{Cl}(\Sigma'_1 \bigcup \{1\}) \text{ and } R_2[p] \in \mathsf{Cl}(\Sigma'_2 \bigcup \{1\}),\, \forall\, p \in \Sigma_1 \times \Sigma_2\Big\}$$

and

$$\mathcal{L}_1 \circledast \mathcal{L}_2 := \Big\{\bigcap \omega\, ;\, \omega \subseteq \Sigma'_{\circledast} \bigcup \{\Sigma_1 \times \Sigma_2\}\Big\}\,,$$

ordered by set-inclusion.

Remark 4.4. Note that $\mathcal{L}_1 \circledast 2 \cong \mathcal{L}_1$.

Notation 4.5. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete atomistic lattices, $a_1 \in \mathcal{L}_1$, and $a_2 \in \mathcal{L}_2$. Then we define

$$\begin{aligned} a_1 \circ a_2 &:= \Sigma[a_1] \times \Sigma[a_2]\,, \\ a_1 \Box\, a_2 &:= (a_1 \circ 1) \bigcup (1 \circ a_2) = (\Sigma[a_1] \times \Sigma_2) \bigcup (\Sigma_1 \times \Sigma[a_2])\,. \end{aligned}$$

Lemma 4.6. *Let $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$ and $\Sigma'_{\oslash} := \{x_1 \Box\, x_2\,;\, (x_1, x_2) \in \Sigma'_1 \times \Sigma'_2\}$. Then $\Sigma'_{\oslash} \subseteq \Sigma'_{\circledast}$ and $\Sigma'_{\oslash}$ is a set of coatoms of $\mathcal{L}_1 \circledast \mathcal{L}_2$.*

Proof. Let $(x_1, x_2) \in \Sigma'_1 \times \Sigma'_2$, $X = x_1 \Box x_2$, and $p \in \Sigma_1 \times \Sigma_2$. Then, $X_1[p] = \Sigma_1$ if $p_2 \leq x_2$ and $X_1[p] = \Sigma[x_1]$ otherwise. Similarly, $X_2[p] = \Sigma_2$ if $p_1 \leq x_1$ and $X_2[p] = \Sigma[x_2]$ otherwise. As a consequence, $X \in \Sigma'_{\circledast}$, thus $X \in \mathcal{L}_1 \circledast \mathcal{L}_2$.

Let $R \in \mathcal{L}_1 \circledast \mathcal{L}_2$ such that $X \subseteq R$ and $X \neq R$. Let $p \in R \backslash X$. Then, $\{p_1\} \times (\{p_2\} \bigcup \Sigma[x_2]) \subseteq R$, hence, since x_2 is a coatom of $\mathcal{L}_2$, by Definition 4.3, $\{p_1\} \times \Sigma_2 \subseteq R$. As a consequence, $(\Sigma[x_1] \bigcup \{p_1\}) \times \Sigma_2 \subseteq R$ (and $\Sigma_1 \times (\Sigma[x_2] \bigcup \{p_2\}) \subseteq R$). Hence, for all $s \in \Sigma_2$, $(\Sigma[x_1] \bigcup \{p_1\}) \times \{s\} \subseteq R_1[s] \in \mathsf{Cl}(\Sigma'_1 \bigcup \{1\})$, thus, since x_1 is a coatom of $\mathcal{L}_1$, $R_1[s] = \Sigma_1$ for all $s \in \Sigma_2$, that is, $R = \Sigma_1 \times \Sigma_2$. As a consequence, X is a coatom of $\mathcal{L}_1 \circledast \mathcal{L}_2$. $\square$

Lemma 4.7. *Let $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$. Then $\mathcal{L}_1 \circledast \mathcal{L}_2$ is a simple closure space on $\Sigma_1 \times \Sigma_2$. Moreover, $\mathcal{L}_1 \circledast \mathcal{L}_2 \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$ and the set of coatoms of $\mathcal{L}_1 \circledast \mathcal{L}_2$ is given by $\Sigma'_{\circledast}$.*

Proof. We first prove that $\mathcal{L}_1 \circledast \mathcal{L}_2$ is a simple closure space on $\Sigma_1 \times \Sigma_2$. By Remark 4.4, we can assume that $\mathcal{L}_1 \neq 2$ and that $\mathcal{L}_2 \neq 2$. By definition, $\mathcal{L}_1 \circledast \mathcal{L}_2$ contains $\Sigma_1 \times \Sigma_2$ and all set-intersections. Let $p \in \Sigma_1 \times \Sigma_2$, and

$$\begin{aligned} X_1 &:= \bigcap \{x_1 \Box\, x_2\,;\, (x_1, x_2) \in \Sigma'[p_1] \times \Sigma'_2\}\ , \\ X_2 &:= \bigcap \{x_1 \Box\, x_2\,;\, (x_1, x_2) \in \Sigma'_1 \times \Sigma'[p_2]\}\ . \end{aligned}$$

Then $p \in X_1 \bigcap X_2$. By Lemma 4.6, X_1, $X_2 \in \mathcal{L}_1 \circledast \mathcal{L}_2$. Moreover

$$X_1 = \bigcup \left\{ \left(\bigcap f^{-1}(1) \right) \times \left(\bigcap f^{-1}(2) \right)\ ;\ f \in 2^{\mathsf{Cl}(\Sigma'[p_1]) \times \mathsf{Cl}(\Sigma'_2)} \right\}\ .$$

Now, if $f^{-1}(1) \neq \mathsf{Cl}(\Sigma'[p_1])$, then $\bigcap f^{-1}(2) = \emptyset$. Therefore, $X_1 = p_1 \circ 1$, and for the same reason, $X_2 = 1 \circ p_2$. Hence $\{p\} = X_1 \bigcap X_2$, thus $\{p\} \in \mathcal{L}_1 \circledast \mathcal{L}_2$. As a consequence, $\mathcal{L}_1 \circledast \mathcal{L}_2$ is a simple closure space on $\Sigma_1 \times \Sigma_2$.

To prove that the set of coatoms of $\mathcal{L}_1 \circledast \mathcal{L}_2$ is given by $\Sigma'_{\circledast}$, it suffices to check that if x, $y \in \Sigma'_{\circledast} \bigcup \{1 \circ 1\}$ and $x \subsetneqq y$, then $y = 1 \circ 1$. Let $p \in \Sigma_1 \times \Sigma_2$ such that $x_2[p] \subsetneqq y_2[p]$. Then $y_2[p] = \Sigma_2$, since by definition, $\bigvee x_2[p]$ is either a coatom of $\mathcal{L}_2$ or 1. Let $q_2 \not\in x_2[p]$. First, since $x \subseteq y$, we have $x_1[q_2] \subseteq y_1[q_2]$. Now, by hypothesis, $p_1 \not\in x_1[q_2]$ whereas p_1 is in $y_1[q_2]$. As a consequence, $y_1[q_2] = \Sigma_1$, for any $q_2 \not\in x_2[p]$, therefore $\Sigma_1 \times (\Sigma_2 \backslash x_2[p]) \subseteq y$. By Axiom $\mathbf{A}_0$, it follows that $\bigvee(\Sigma_2 \backslash x_2[p]) = 1$. Thus, we find that $y = 1 \circ 1$.

We now check that Axiom $\mathbf{A}_0$ holds in $\mathcal{L}_1 \circledast \mathcal{L}_2$. Let x, $y \in \Sigma'_{\circledast}$ and

$$\begin{aligned} A &:= \{p_1 \in \Sigma_1\,;\, x_2[p_1] = \Sigma_2\} \\ B &:= \{p_1 \in \Sigma_1\,;\, y_2[p_1] = \Sigma_2\}\,. \end{aligned}$$

Note that

$$A = \bigcap \{x_1[s]\,;\, s \in \Sigma_2\} \qquad \text{and} \qquad B = \bigcap \{y_1[s]\,;\, s \in \Sigma_2\}\,,$$

hence A, $B \in \mathsf{Cl}(\mathcal{L}_1)$. Indeed,

$$r \in A \Leftrightarrow x_2[r] = \Sigma_2 \Leftrightarrow r \times \Sigma_2 \subseteq x \Leftrightarrow r \in x_1[s],\ \forall s \in \Sigma_2 \Leftrightarrow r \in \bigcap\{x_1[s]\,;\, s \in \Sigma_2\}\,.$$

Suppose now that $x \bigcup y = \Sigma_1 \times \Sigma_2$. Then, for any $p \in \Sigma_1 \times \Sigma_2$, $x_2[p] \cup y_2[p] = \Sigma_2$. As a consequence, since Axiom $\mathbf{A}_0$ holds in $\mathcal{L}_2$, we have that $x_2[p] \neq \Sigma_2 \Rightarrow y_2[p] = \Sigma_2$, and $y_2[p] \neq \Sigma_2 \Rightarrow x_2[p] = \Sigma_2$; whence $A \bigcup B = \Sigma_1$, a contradiction, since Axiom $\mathbf{A}_0$ holds in $\mathcal{L}_1$.

It remains to check that Axiom $\mathbf{A}_0$ holds in $(\mathcal{L}_1 \circledast \mathcal{L}_2)^{\mathrm{op}}$. Let p, $q \in \Sigma_1 \times \Sigma_2$. Since Axiom $\mathbf{A}_0$ holds in $\mathcal{L}_1^{\mathrm{op}}$ and in $\mathcal{L}_2^{\mathrm{op}}$, there is $(x_1, x_2) \in \Sigma'_1 \times \Sigma'_2$ such that for $i = 1$ and $i = 2$, $p_i \bigwedge x_i = q_i \bigwedge x_i = 0$. As a consequence, we have $p \bigwedge (x_1 \Box x_2) = q \bigwedge (x_1 \Box x_2) = 0$. Moreover, by Lemma 4.6, $x_1 \Box x_2$ is a coatom of $\mathcal{L}_1 \circledast \mathcal{L}_2$. $\Box$

Lemma 4.8. *Let $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$. There is a bijection*

$$\xi : \Sigma'_{\circledast} \bigcup \{1_{\circledast}\} \to \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_1, \mathcal{L}_2^{\mathrm{op}})\,,$$

such that for all $x \in \Sigma'_{\circledast} \bigcup \{1_{\circledast}\}$, we have $x = \{p \circ \xi(x)(p)\,;\, p \in \Sigma_1\}$.

Proof. Let $x \in \Sigma'_{\circledast} \bigcup \{1_{\circledast}\}$. Define $F_x : \Sigma_1 \bigcup \{0\} \to \Sigma'_2 \bigcup \{1\}$ as

$$F_x(p_1) := \bigvee x_2[p_1]\,,$$

and $F_x(0) = 1$. Moreover, define $f_x : \mathcal{L}_1 \to \mathcal{L}_2^{\mathrm{op}}$ as $f_x(a) = \bigvee_{\mathrm{op}} F_x(\Sigma[a])$. Obviously, f_x sends atoms to atoms or 0. Let $b \in \mathcal{L}_2^{\mathrm{op}}$ and $A := F_x^{-1}(\Sigma'[b] \bigcup \{1\})$ (*i.e.* $A = \{r \in \Sigma_1\,;\, F_x(r) \leq_{\mathrm{op}} b\}$). Note that

$$\begin{aligned} r \in A \Leftrightarrow \Sigma[b] \subseteq x_2[r] \Leftrightarrow \{r\} \times \Sigma[b] \subseteq x \Leftrightarrow (r, s) \in x\,,\ \forall s \in \Sigma[b] \\ \Leftrightarrow r \in x_1[s]\,,\ \forall s \in \Sigma[b] \Leftrightarrow r \in \bigcap\{x_1[s]\,;\, s \in \Sigma[b]\}\,, \end{aligned}$$

hence $A = \bigcap\{x_1[s]\,;\, s \in \Sigma[b]\}$. As a consequence, $A \in \mathsf{Cl}(\mathcal{L}_1)$, therefore, by Lemma 3.3, f_x preserves arbitrary joins.

Let q be an atom of $\mathcal{L}_2$. Then

$$f_x^{\circ}(q) = \bigvee\{a \in \mathcal{L}_1\,;\, f_x(a) \leq_{\mathrm{op}} q\} = \bigvee\{p \in \Sigma_1\,;\, q \leq f_x(p)\} = \bigvee x_1[q]\,.$$

Therefore, $f_x \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_1, \mathcal{L}_2^{\mathrm{op}})$.

Let $f \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_1, \mathcal{L}_2^{\mathrm{op}})$. Define $x^f \subseteq \Sigma_1 \times \Sigma_2$ as $x^f = \bigcup\{p \circ f(p)\,;\, p \in \Sigma_1\}$. Let $p \in \Sigma_1 \times \Sigma_2$. Then

$$x_2^f[p] = \Sigma[f(p_1)] \in \mathsf{Cl}(\Sigma'_2 \bigcup \{1\})\,,$$

and by Lemma 3.3

$$\begin{aligned} x_1^f[p] &= \{r \in \Sigma_1\,;\, p_2 \leq f(r)\} = \{r \in \Sigma_1\,;\, f(r) \leq_{\mathrm{op}} p_2\} \\ &= \Sigma[\bigvee\{r \in \Sigma_1\,;\, f(r) \leq_{\mathrm{op}} p_2\}] = \Sigma[\bigvee\{a \in \mathcal{L}_1\,;\, f(a) \leq_{\mathrm{op}} p_2\}] \\ &= \Sigma[f^{\circ}(p_2)] \in \mathsf{Cl}(\Sigma'_1 \bigcup \{1\})\,. \end{aligned}$$

Obviously, we have $x^{f_x} = x$ and $f_{x^f} = f$. $\Box$

Lemma 4.9. *For i, $j \in \{1, 2\}$, let $\mathcal{L}_i^j \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$ and $f_i \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_i^1, \mathcal{L}_i^2)$. Then there is a unique $u \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_1^1 \circledast \mathcal{L}_2^1, \mathcal{L}_1^2 \circledast \mathcal{L}_2^2)$, with $u(p) = (f_1(p_1), f_2(p_2))$, for any $p \in \Sigma_1^1 \times \Sigma_2^1$ such that $f_1(p_1) \neq 0$ and $f_2(p_2) \neq 0$. We denote u by $f_1 \circledast f_2$.*

Proof. Write F_i for f_i restricted to atoms and define F as $F(p_1,p_2)=0_\circledast$ if $F_1(p_1)=0$ or $F_2(p_2)=0$, and $F(p_1,p_2)=(F_1(p_1),F_2(p_2))$ otherwise. Define u as

$$u(a)=\bigvee\{F(p_1,p_2)\,;\,(p_1,p_2)\in a\}$$

By Lemma 3.3, it suffices to show that for any coatom x of $\mathcal{L}_1^2\circledast\mathcal{L}_2^2$, $F^{-1}(x\bigcup\{0_\circledast\})$ is a coatom of $\mathcal{L}_1^1\circledast\mathcal{L}_2^1$ or 1. Let $p\in\Sigma_1\times\Sigma_2$. Suppose that $f_2(p_2)\neq 0$. Then,

$$\begin{aligned}(F^{-1}(x\textstyle\bigcup\{0_\circledast\}))_1[p]&=\{q_1\in\Sigma_1\,;\,(q_1,p_2)\in F^{-1}(x\textstyle\bigcup\{0_\circledast\})\}\\&=\{q_1\in\Sigma_1\,;\,F(q_1,p_2)\in x\textstyle\bigcup\{0_\circledast\}\}\\&=\{q_1\in\Sigma_1\,;\,F_1(q_1)\neq 0\text{ and }(F_1(q_1),F_2(p_2))\in x\}\textstyle\bigcup F_1^{-1}(0)\\&=F_1^{-1}\left(\{r\in\Sigma_1\,;\,(r,F_2(p_2))\in x\}\textstyle\bigcup\{0\}\right)\\&=F_1^{-1}(x_1[F_2(p_2)]\textstyle\bigcup\{0\})\ .\end{aligned}$$

On the other hand, if $f_2(p_2)=0$, then $(F^{-1}(x\bigcup\{0_\circledast\}))_1[p]=\Sigma_1$. As a consequence, we find that for $i=1$ and $i=2$, $(F^{-1}(x\bigcup\{0_\circledast\}))_i[p]\in\mathsf{Cl}(\Sigma_i'\bigcup\{1\})$ for any $p\in\Sigma_1^1\times\Sigma_2^1$. □

Proposition 4.10. $-\circledast-:\mathbf{Cal^0_{Sym}}\times\mathbf{Cal^0_{Sym}}\to\mathbf{Cal^0_{Sym}}$ *is a bifunctor.*

5. $*-$autonomous structure on $\mathbf{Cal^0_{Sym}}$

We can now turn to our main result. The functor given in the following definition explains where the tensor product of $\mathbf{Cal^0_{Sym}}$ comes from. It is also useful to understand the $*-$autonomous structure of the category. By Lemma 3.3, the functor given in the following definition is well-defined.

Definition 5.1. Let $\mathcal{F}:\mathbf{Cal^0_{Sym}}\to\mathbf{Chu}_{2_0}$ be the functor defined on objects as $\mathcal{F}(\mathcal{L})=(\Sigma\bigcup\{0\},r,\Sigma'\bigcup\{1\})$, with $\Sigma\bigcup\{0\}$ pointed by 0 and $\Sigma'\bigcup\{1\}$ pointed by 1, and with $r(p,x)=0\Leftrightarrow p\leq x$ for any $p\in\Sigma\bigcup\{0\}$ and $x\in\Sigma'\bigcup\{1\}$ (hence $r(0,\cdot)\equiv r(\cdot,1)\equiv 0$). On arrows, the functor $\mathcal{F}$ is defined as $\mathcal{F}(f)=(f,f^\circ)$.

Lemma 5.2. *The functor* $\mathcal{F}:\mathbf{Cal^0_{Sym}}\to\mathbf{Chu}_{2_0}$ *is full and faithful. In particular,* $\mathcal{F}(\mathcal{L}_1)\cong\mathcal{F}(\mathcal{L}_2)\implies\mathcal{L}_1\cong\mathcal{L}_2$.

Proof. Let $\mathcal{L}_1$, $\mathcal{L}_2\in\mathbf{Cal^0_{Sym}}$. To prove that $\mathcal{F}$ is faithful, let $f,g\in\mathbf{Cal^0_{Sym}}(\mathcal{L}_1,\mathcal{L}_2)$ be such that $\mathcal{F}(f)=\mathcal{F}(g)$. Thus, $f=g$ on atoms, and since those maps preserve arbitrary joins, for any $a\in\mathcal{L}_1$ we have $f(a)=\bigvee f(\Sigma[a])=\bigvee g(\Sigma[a])=g(a)$.

To show that $\mathcal{F}$ is full, let $(f,g):\mathcal{F}(\mathcal{L}_1)\to\mathcal{F}(\mathcal{L}_2)$ be an arrow of $\mathbf{Chu}_{2_0}$. Write $\mathcal{F}(\mathcal{L}_i)$ as $(\Sigma_i\bigcup\{0\},r_i,\Sigma_i'\bigcup\{1\})$. Define $h:\mathcal{L}_1\to\mathcal{L}_2$ as $h(a)=\bigvee f(\Sigma[a])$. Since f is a pointed map from $\Sigma_1\bigcup\{0_1\}$ to $\Sigma_2\bigcup\{0_2\}$, we have that h preserves 0 and sends atoms to atoms or 0. Denote by H the restriction of h to atoms (hence $H=f$). Let x be a coatom of $\mathcal{L}_2$. Then

$$\begin{aligned}H^{-1}(\Sigma[x]\textstyle\bigcup\{0\})&=\{p\in\Sigma_1\,;\,r_2(f(p),x)=0\}\\&=\{p\in\Sigma_1\,;\,r_1(p,g(x))=0\}=\Sigma[g(x)]\,.\end{aligned}$$

Therefore, by Lemma 3.3, h preserves arbitrary joins, and moreover, we find that $h\in\mathbf{Cal^0_{Sym}}(\mathcal{L}_1,\mathcal{L}_2)$.

Finally, if $\mathcal{F}(\mathcal{L}_1)\cong\mathcal{F}(\mathcal{L}_2)$, then $\mathsf{Cl}(\mathcal{L}_1)=\mathsf{Cl}(\mathcal{L}_2)$, therefore $\mathcal{L}_1\cong\mathcal{L}_2$. □

Lemma 5.3. *Let* $\mathcal{L}$, $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$ *and* $f \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_1, \mathcal{L}_2)$*. We have that* $\mathcal{F}(\mathcal{L}^{\mathrm{op}}) = \mathcal{F}(\mathcal{L})^{\perp}$ *and* $\mathcal{F}(f^{\mathrm{op}}) = \mathcal{F}(f)^{\perp}$*, so that* $\mathcal{F}(\mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}})$ *is closed under* $\perp$*. Moreover,* $\mathcal{F}(2) \cong \top$*.*

Proof. By definition,

$$\mathcal{F}(\mathcal{L}^{\mathrm{op}}) = (\Sigma' \bigcup \{1\}, r^{\mathrm{op}}, \Sigma \bigcup \{0\}) \,,$$

with $r^{\mathrm{op}}(1, \cdot) \equiv 0 \equiv r^{\mathrm{op}}(\cdot, 0)$, and for any $x \in \Sigma'$ and $p \in \Sigma$, $r^{\mathrm{op}}(x, p) = 0 \Leftrightarrow x \leq_{\mathrm{op}} p \Leftrightarrow p \leq x$. As a consequence, $r^{\mathrm{op}} = \check{r}$ and $\mathcal{F}(\mathcal{L}^{\mathrm{op}}) = \mathcal{F}(\mathcal{L})^{\perp}$.

Moreover, $\mathcal{F}(f^{\mathrm{op}}) = (f^{\mathrm{op}}, (f^{\mathrm{op}})^{\circ})$. Now, $f^{\mathrm{op}} : \mathcal{L}_2^{\mathrm{op}} \to \mathcal{L}_1^{\mathrm{op}}$, so that $(f^{\mathrm{op}})^{\circ} : \mathcal{L}_1^{\mathrm{op}} \to \mathcal{L}_2^{\mathrm{op}}$. Let p be an atom of $\mathcal{L}_1$. Then

$$\begin{aligned} (f^{\mathrm{op}})^{\circ}(p) &= \bigvee\nolimits_{\mathrm{op}} \{b \in \mathcal{L}_2^{\mathrm{op}} \,;\, f^{\mathrm{op}}(b) \leq_{\mathrm{op}} p\} \\ &= \bigwedge \{b \in \mathcal{L}_2 \,;\, p \leq f^{\circ}(b)\} = \bigwedge \{b \in \mathcal{L}_2 \,;\, f(p) \leq b\} = f(p) \,. \end{aligned}$$

Recall that by definition, for any $b \in \mathcal{L}_2^{\mathrm{op}}$, we have $f^{\mathrm{op}}(b) = f^{\circ}(b)$. As a consequence, $\mathcal{F}(f^{\mathrm{op}}) = (f^{\circ}, f) = \mathcal{F}(f)^{\perp}$.

By definition, $\mathcal{F}(2) = (A, r, X)$ with $A = \{1, 0\}$ pointed by 0 and $X = \{0, 1\}$ pointed by 1, and with $r(0, \cdot) \equiv 0 \equiv r(\cdot, 1)$, and $r(1, 0) = 1$. As a consequence, $\mathcal{F}(2) \cong \top$. □

Lemma 5.4. *Let* $\mathbf{C}$ *be the subcategory of* $\mathbf{Chu}_{2_0}$ *formed by closing* $\mathcal{F}(\mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}})$ *under isomorphisms. Then* $- \otimes_C -$ *is a bifunctor in* $\mathbf{C}$*. Moreover, there are isomorphisms*

$$\alpha_{\mathcal{L}_1 \mathcal{L}_2} : \mathcal{F}(\mathcal{L}_1) \otimes_C \mathcal{F}(\mathcal{L}_2) \to \mathcal{F}(\mathcal{L}_1 \circledast \mathcal{L}_2) \,,$$

natural for all objects $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *of* $\mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$*.*

Proof. Let $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$. By Definition 2.9

$$\mathcal{F}(\mathcal{L}_1) \otimes_C \mathcal{F}(\mathcal{L}_2) = ((\Sigma_1 \bigcup \{0\}) \sharp (\Sigma_2 \bigcup \{0\}), t, \mathbf{Chu}_{2_0}(\mathcal{F}(\mathcal{L}_1), \mathcal{F}(\mathcal{L}_2)^{\top})) \,,$$

and by Lemma 4.7,

$$\mathcal{F}(\mathcal{L}_1 \circledast \mathcal{L}_2) = ((\Sigma_1 \times \Sigma_2) \bigcup \{0\}, t', \Sigma'_{\circledast} \bigcup \{1\}) \,,$$

where $t'(p, x) = 0 \Leftrightarrow p \in x$, for any $p \in \Sigma_1 \times \Sigma_2$ and $x \in \Sigma'_{\circledast}$.

Define

$$\chi : (\Sigma_1 \bigcup \{0\}) \sharp (\Sigma_2 \bigcup \{0\}) \to (\Sigma_1 \times \Sigma_2) \bigcup \{0\};\, \chi(0_{\sharp}) = 0,\, \chi(p_1, p_2) = (p_1, p_2) \,,$$

and let ξ be the bijection of Lemma 4.8 (write $\xi(x) = f_x$). Moreover, define $\alpha := (\chi, \mathcal{F} \circ \xi)$. By definition of the smash product, the map χ is a bijection.

Write $\mathcal{F}(\mathcal{L}_i)$ as $(\Sigma_i \bigcup \{0\}, r_i, \Sigma'_i \bigcup \{1\})$ and let $p \in \Sigma_1 \times \Sigma_2$ and $x \in \Sigma'_{\circledast}$. Then, we have

$$\begin{aligned} t'(\chi(p), x) = 0 &\Leftrightarrow p_2 \in x_2[p_1] \Leftrightarrow p_2 \leq f_x(p_1) \Leftrightarrow r_2(p_2, f_x(p_1)) = 0 \\ &\Leftrightarrow t((p_1, p_2), \mathcal{F}(f_x)) = 0 \,. \end{aligned}$$

Let $(f, g) \in \mathbf{Chu}_{2_0}(\mathcal{F}(\mathcal{L}_1), \mathcal{F}(\mathcal{L}_2)^{\perp})$ and $h \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_1, \mathcal{L}_2^{\mathrm{op}})$ with $\mathcal{F}(h) = (f, g)$. Then

$$t(\chi^{-1}(p), (f, g)) = 0 \Leftrightarrow r_2(p_2, f(p_1)) = 0 \Leftrightarrow r_2(p_2, h(p_1)) = 0 \Leftrightarrow t'(p, \xi^{-1}(h)) = 0 \,.$$

As a consequence, we find that $\alpha : \mathcal{F}(\mathcal{L}_1) \otimes_C \mathcal{F}(\mathcal{L}_2) \to \mathcal{F}(\mathcal{L}_1 \circledast \mathcal{L}_2)$ is an invertible arrow of $\mathbf{Chu}_{2_0}$.

Finally, we prove that

$$\mathcal{F}(f_1 \circledast f_2) \circ \alpha = \alpha \circ (\mathcal{F}(f_1) \otimes_C \mathcal{F}(f_2)) .$$

Let $\mathcal{L}_1^2$, $\mathcal{L}_2^2 \in \mathbf{Cal^0_{Sym}}$, $f_1 \in \mathbf{Cal^0_{Sym}}(\mathcal{L}_1, \mathcal{L}_1^2)$ and $f_2 \in \mathbf{Cal^0_{Sym}}(\mathcal{L}_2, \mathcal{L}_2^2)$. Moreover, let $p = (p_1, p_2) \in \Sigma_1 \times \Sigma_2$ and x be a coatom of $\mathcal{L}_1^2 \circledast \mathcal{L}_2^2$. Then

$$\bigvee ((f_1 \circledast f_2)^\circ(x))_i[p] = f_i^\circ(\bigvee x_i[f(p)])$$

(see the proof of Lemma 4.9). Therefore, we find that

$$\begin{aligned}(\mathcal{F} \circ \xi \circ (f_1 \circledast f_2)^\circ)(x) &= (f_2^\circ \circ f_x \circ f_1, (f_2^\circ \circ f_x \circ f_1)^\circ) \\ &= (f_2^\circ \circ f_x \circ f_1, f_1^\circ \circ f_x^\circ \circ f_2) = (\mathcal{F}(f_1) \otimes_C \mathcal{F}(f_2))((\mathcal{F} \circ \xi)(x)) .\end{aligned}$$

□

Theorem 5.5. *The category* $\langle \mathbf{Cal^0_{Sym}}, \circledast, 2, \multimap, 2 \rangle$, *with* $\mathcal{L}_1 \multimap \mathcal{L}_2 = (\mathcal{L}_1 \circledast \mathcal{L}_2^{\mathrm{op}})^{\mathrm{op}}$, *is* $*$*-autonomous.*

Proof. This theorem can be proved directly, however it also follows easily from Proposition 4.10, the Lemmata 5.2, 5.3, 5.4, and Proposition 2.12. □

Remark 5.6. We now give two examples where Axiom $\mathbf{A}_0$ does not hold in $\mathcal{L}_1$ and $\mathcal{L}_2$, and where the set $\Sigma'_\circledast$ defined in Definition 4.3 is not a set of coatoms. More precisely, $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$ and there is R, $S \in \Sigma'_\circledast$ with R a proper subset of S and $S \neq \Sigma_1 \times \Sigma_2$.

Now, in the proof of Lemma 5.4, we have shown that $\mathcal{F}(\mathcal{L}_1) \otimes_C \mathcal{F}(\mathcal{L}_2) \cong ((\Sigma_1 \times \Sigma_2) \bigcup \{0\}, t', \Sigma'_\circledast \bigcup \{1\})$. As a consequence, since $R \underset{\neq}{\subsetneq} S \underset{\neq}{\subsetneq} \Sigma_1 \times \Sigma_2$, there is no complete atomistic coatomistic lattice $\mathcal{L}_0$ such that $\mathcal{F}(\mathcal{L}_0) \cong \mathcal{F}(\mathcal{L}_1) \otimes_C \mathcal{F}(\mathcal{L}_2)$.

In Example 5.7 $\mathcal{L}$ is a powerset lattice whereas in Example 5.8 $\mathcal{L}$ is an irreducible complete atomistic orthocomplemented lattice. Recall that Axiom $\mathbf{A}_0$ implies irreducibility.

Example 5.7. Let $\mathcal{L} = 2^\Sigma$ be a powerset lattice and $r_0 \in \Sigma$ (with $\#\Sigma \geq 2$). Let $R = \bigcup \{\{r\} \times (\Sigma \backslash \{r\}) \, ; \, r \in \Sigma\}$ and $S = R \bigcup \{r_0\} \times \Sigma$. Then R and S are subsets of $\Sigma \times \Sigma$, and obviously, for all $p \in \Sigma \times \Sigma$, $R_1[p]$, $R_2[p]$, $S_1[p]$, and $S_2[p]$ are in $\mathsf{Cl}(\Sigma' \bigcup \{1\})$. Moreover, R is a proper subset of S, and $S \neq \Sigma \times \Sigma$.

Example 5.8. Let $\mathcal{L}$ be the orthocomplemented simple closure space on $\Sigma = \mathbb{Z}_6$ with $n' = \{(n+2), (n+3), (n+4)\}$, where $(m) := m \bmod 6$ and $'$ denotes the orthocomplementation. Use the map $g : \Sigma \to \mathbb{C}; \, n \mapsto \mathrm{e}^{\mathrm{i}n\pi/3}$ to check that $n \perp m \Leftrightarrow n \in m'$ is indeed symmetric, anti-reflexive and separating, *i.e.* for any p, $q \in \Sigma$ there is $r \in \Sigma$ such that $p \perp r$ and $q \not\perp r$. Obviously, $\mathcal{L}$ is irreducible, but $\mathcal{L}$ does not satisfy Axiom $\mathbf{A}_0$. For instance, $0' \bigcup 3' = \Sigma$.

Let $R \subseteq \Sigma \times \Sigma$ defined as $R = \{0,1,2\} \times \{0,1,2\} \bigcup \{3,4,5\} \times \{3,4,5\}$ and $S := 4' \times \Sigma \bigcup \Sigma \times 1'$. Obviously, R and S are in $\Sigma'_\circledast$. Moreover, R is a proper subset of S, and $S \neq \Sigma \times \Sigma$.

6. Comparison of $\circledast$ with other tensor products

In this section, we compare the tensor product $\circledast$ with the *separated product* of Aerts [1], the box product of Grätzer and Wehrung [11], and with the tensor product of Shmuely [16].

Definition 6.1. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete atomistic lattices. Write $\mathsf{Aut}(\mathcal{L}_i)$ for the group of automorphisms of $\mathcal{L}_i$. Define

$$\mathcal{L}_1 \owedge \mathcal{L}_2 := \left\{ \bigcap \omega \, ; \, \omega \subseteq \{a_1 \square a_2 \, ; \, a \in \mathcal{L}_1 \times \mathcal{L}_2\} \right\} ,$$

$$\mathcal{L}_1 \ovee \mathcal{L}_2 := \{ R \subseteq \Sigma_1 \times \Sigma_2 \, ; \, R_1[p] \in \mathsf{Cl}(\mathcal{L}_1) \text{ and } R_2[p] \in \mathsf{Cl}(\mathcal{L}_2), \forall \, p \in \Sigma_1 \times \Sigma_2 \} .$$

ordered by set-inclusion.

Moreover, $\mathcal{S}(\mathcal{L}_1, \mathcal{L}_2)$ is defined as the set of all simple closure spaces $\mathcal{L}$ on $\Sigma_1 \times \Sigma_2$ such that

(1) $\mathcal{L}_1 \owedge \mathcal{L}_2 \subseteq \mathcal{L} \subseteq \mathcal{L}_1 \ovee \mathcal{L}_2$, and
(2) for all $(u_1, u_2) \in \mathsf{Aut}(\mathcal{L}_1) \times \mathsf{Aut}(\mathcal{L}_2)$, there is $u \in \mathsf{Aut}(\mathcal{L})$ such that $u(p_1, p_2) = (u_1(p_1), u_2(p_2))$, for all atoms $p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2$.

Remark 6.2. By Lemma 4.6 and from the proof of Lemma 4.7, it follows that if $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$, then $\Sigma'_{\owedge} \subseteq \Sigma'_{\circledast} \subseteq \Sigma'_{\ovee}$, where $\Sigma'_{\ovee}$ denotes the set of coatoms of $\mathcal{L}_1 \ovee \mathcal{L}_2$. Note that obviously, $\Sigma'_{\owedge}$ is the set of coatoms of $\mathcal{L}_1 \owedge \mathcal{L}_2$. Moreover, $\mathcal{L}_1 \owedge \mathcal{L}_2$ is coatomistic (see [12]).

Theorem 6.3. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete atomistic lattices. Then $\mathcal{L}_1 \owedge \mathcal{L}_2$ and $\mathcal{L}_1 \ovee \mathcal{L}_2$ are simple closure spaces on $\Sigma_1 \times \Sigma_2$. Moreover, $\mathcal{S}(\mathcal{L}_1, \mathcal{L}_2)$ (ordered by set-inclusion) is a complete lattice.*

Proof. See [12]. □

Remark 6.4. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are orthocomplemented, so is $\mathcal{L}_1 \owedge \mathcal{L}_2$; it is the *separated product* of Aerts [1] (see [12]). The binary relation on $\Sigma_1 \times \Sigma_2$ defined by $p \# q \Leftrightarrow p_1 \perp_1 q_1$ or $p_2 \perp_2 q_2$, induces an orthocomplementation of $\mathcal{L}_1 \owedge \mathcal{L}_2$.

For atomistic lattices, define $\mathcal{L}_1 \owedge_n \mathcal{L}_2$ by taking only finite intersections in Definition 6.1. Then $\mathcal{L}_1 \owedge_n \mathcal{L}_2 \cong \mathcal{L}_1 \square \mathcal{L}_2$ which is the *box-product* of Grätzer and Wehrung [11] (see [12]).

By Lemma 3.3, the functor given in the following definition is well-defined.

Definition 6.5. Let $\mathbf{Cal}$ be the category of complete atomistic lattices with maps preserving arbitrary joins and sending atoms to atoms. We denote by $\mathcal{G} : \mathbf{Cal} \to \mathbf{Chu}(\mathbf{Set}, 2)$ the functor defined on objects as $\mathcal{G}(\mathcal{L}) := (\Sigma, r, \mathcal{L})$ with $r(p, a) = 1 \Leftrightarrow p \leq a$, and on arrows as $\mathcal{G}(f) = (f, f^{\circ})$.

Proposition 6.6. *The functor $\mathcal{G}$ is full and faithful. Moreover for any $\mathcal{L}_1$, $\mathcal{L}_2 \in$ $\mathbf{Cal}$, we have $\mathcal{G}(\mathcal{L}_1 \ovee \mathcal{L}_2) \cong \mathcal{G}(\mathcal{L}_1) \otimes_C \mathcal{G}(\mathcal{L}_2)$, where $- \otimes_C -$ is the bifunctor in the category $\mathbf{Chu}(\mathbf{Set}, 2)$.*

Proof. To prove that $\mathcal{G}$ is full and faithful, we can proceed as in the proof of Lemma 5.2. For the rest of the proof, we refer to [12]. □

Remark 6.7. For $\mathcal{L}_1$ and $\mathcal{L}_2$ complete atomistic lattices, we have $\mathcal{L}_1 \ovee \mathcal{L}_2 \cong \mathcal{L}_1 \otimes \mathcal{L}_2$ the tensor product of Shmuely [16] (see [12]).

Theorem 6.8. *Let $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$. Then $\mathcal{L}_1 \circledast \mathcal{L}_2 \in \mathcal{S}(\mathcal{L}_1, \mathcal{L}_2)$.*

Proof. By definition 4.3, $\Sigma'_{\circledast} \subseteq \mathcal{L}_1 \ovee \mathcal{L}_2$. Now, by Theorem 6.3, $\mathcal{L}_1 \ovee \mathcal{L}_2$ is a simple closure space on $\Sigma_1 \times \Sigma_2$. As a consequence, from Definition 4.3 it follows that $\mathcal{L}_1 \circledast \mathcal{L}_2 \subseteq \mathcal{L}_1 \ovee \mathcal{L}_2$.

Similarly, by Lemma 4.6, $\Sigma'_{\owedge} \subseteq \Sigma'_{\circledast}$, hence by Remark 6.2 and Definitions 4.3 and 6.1, it follows that $\mathcal{L}_1 \owedge \mathcal{L}_2 \subseteq \mathcal{L}_1 \circledast \mathcal{L}_2$.

Finally, by Lemma 4.9, for all automorphisms u_1 and u_2, $u_1 \circledast u_2 \in \mathsf{Aut}(\mathcal{L}_1 \circledast \mathcal{L}_2)$. $\square$

We end this section with two examples.

Example 6.9. As in Example 5.8, let $\mathcal{L}$ be the orthocomplemented simple closure space on $\Sigma = \mathbb{Z}_{12}$ with $n' = \{(n+5), (n+6), (n+7)\}$, where $(m) := m \bmod 12$. Note that Axiom $\mathbf{A}_0$ holds in $\mathcal{L}$ but that $\mathcal{L}$ does not have the covering property since $2 \bigvee 8 = \Sigma \gneqq 2' \gneqq 8$.

Let

$$x := (0' \times 0') \bigcup (3' \times 3') \bigcup (6' \times 6') \bigcup (9' \times 9')\,.$$

By Definition 4.3, $x \in \mathcal{L} \circledast \mathcal{L}$ (note that $x \in \Sigma'_{\circledast}$), whereas obviously, $x^{\#} = \emptyset$ (see Remark 6.4), therefore $x \notin \mathcal{L} \owedge \mathcal{L}$.

Let

$$R := (0' \times 0') \bigcup ((2' \bigcap 3') \times (2' \bigcap 3')) \bigcup (5' \times 5') \bigcup (8' \times 8') \bigcup (4 \times 4)\,.$$

By Definition 6.1, $R \in \mathcal{L} \ovee \mathcal{L}$. **Claim**: R is a coatom of $\mathcal{L} \ovee \mathcal{L}$. [*Proof.* Let $p = (p_1, p_2)$ be an atom not under R. Define $z := p \bigvee R$, where the join is taken in $\mathcal{L} \ovee \mathcal{L}$. If p_1 or p_2 is in $0'$, $5'$ or $8'$, then $p \bigvee R = \Sigma \times \Sigma$. Indeed, suppose for instance that p_1 is in $0'$. Then, $p_1 \times (0' \bigcup \{p_1\}) \subseteq z$, hence $p_1 \times \Sigma \subseteq z$. Therefore,

$$(p_1 \bigvee 5') \times 5' \bigcup (p_1 \bigvee 8') \times 8' \subseteq z\,.$$

Now, since $0'$, $5'$ and $8'$ are disjoint, $p_1 \notin 5'$ and $p_1 \notin 8'$, thus $p_1 \bigvee 5' = \Sigma$ and $p_1 \bigvee 8' = \Sigma$. As a consequence, $\Sigma \times (5' \bigcup 8') \subseteq z$, whence $z = \Sigma \times \Sigma$.

Finally, suppose for instance that $p_1 = 4$ and $p_2 \in 2' \bigcap 3' = \{8, 9\}$. Then, $4 \times (4 \bigvee p_2) = 4 \times \Sigma \subseteq z$, hence

$$(4 \bigvee 0') \times 0' \bigcup (4 \bigvee 5') \times 5' \bigcup (4 \bigvee 8') \times 8' \subseteq z\,,$$

therefore $\Sigma \times (0' \bigcup 5' \bigcup 8') \subseteq z$. As a consequence, $z = \Sigma \times \Sigma$.]

Obviously, $R_2[(4, \cdot)] = \{4\} \notin \Sigma'$, hence R is not a coatom of $\mathcal{L} \circledast \mathcal{L}$. As a consequence, $\mathcal{L} \circledast \mathcal{L} \neq \mathcal{L} \ovee \mathcal{L}$ (see Remark 6.2).

To summarize, we have $\mathcal{L} \owedge \mathcal{L} \subsetneqq \mathcal{L} \circledast \mathcal{L} \subsetneqq \mathcal{L} \ovee \mathcal{L}$.

Example 6.10. We leave it as an exercise to prove that

$$\mathsf{MO}_3 \circledast \mathsf{MO}_4 = \mathsf{MO}_3 \owedge \mathsf{MO}_4\,,$$

where MO_n was defined in the introduction.

7. Weak bimorphisms

We now prove that the tensor product $\circledast$ can be defined as the solution of a universal problem with respect to *weak bimorphisms*.

Definition 7.1. Let $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L} \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$ and $f : \mathcal{L}_1 \times \mathcal{L}_2 \to \mathcal{L}$ be a map. Then f is a *weak bimorphism* if for any $p_1 \in \Sigma_1$ and for any $p_2 \in \Sigma_2$, we have $f(-, p_2) \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_1, \mathcal{L})$ and $f(p_1, -) \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_2, \mathcal{L})$. Moreover, $\mathcal{L}$ is a *w−tensor product* of $\mathcal{L}_1$ and $\mathcal{L}_2$ if there is a weak bimorphism $f : \mathcal{L}_1 \times \mathcal{L}_2 \to \mathcal{L}$ such that for any

$\mathcal{L}_0 \in \mathbf{Cal^0_{Sym}}$ and any weak bimorphism $g : \mathcal{L}_1 \times \mathcal{L}_2 \to \mathcal{L}_0$, there is a unique arrow $h \in \mathbf{Cal^0_{Sym}}(\mathcal{L}, \mathcal{L}_0)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{L}_1 \times \mathcal{L}_2 \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{L} \arrow[dl, "!h"] \\
\mathcal{L}_0 &
\end{tikzcd}$$

Remark 7.2. By definition, the $w-$tensor product is unique up to isomorphisms.

Theorem 7.3. *Let $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathbf{Cal^0_{Sym}}$. Then $\mathcal{L}_1 \circledast \mathcal{L}_2$ is the $w-$tensor product of $\mathcal{L}_1$ and $\mathcal{L}_2$.*

Proof. Define $f : \mathcal{L}_1 \times \mathcal{L}_2 \to \mathcal{L}_1 \circledast \mathcal{L}_2$ as $f(a) = \bigvee(a_1 \circ a_2)$ where the join is taken in $\mathcal{L}_1 \circledast \mathcal{L}_2$. Note that

$$a_1 \circ a_2 = (a_1 \circ 1) \bigcap (1 \circ a_2) = (a_1 \square 0) \bigcap (0 \square a_2)\,.$$

Hence, by Lemma 4.6, $f(a) = a_1 \circ a_2$.

Let $p_1 \in \Sigma_1$ and $\omega \subseteq \mathcal{L}_2$. Then, obviously, $f(p_1, x) \subseteq f(p_1, \bigvee \omega)$ for all $x \in \omega$, hence $\bigvee\{f(p_1, x)\,;\, x \in \omega\} \subseteq f(p_1, \bigvee \omega)$. As a consequence, there is $B \subseteq \Sigma_2$ such that $\bigvee\{f(p_1, x)\,;\, x \in \omega\} = p_1 \times B$ with $B \subseteq \Sigma[\bigvee \omega]$ and $\Sigma[x] \subseteq B$ for all $x \in \omega$. Now, since $\mathcal{L}_1 \circledast \mathcal{L}_2 \subseteq \mathcal{L}_1 \circledcirc \mathcal{L}_2$, $B \in \mathsf{Cl}(\mathcal{L}_2)$. Therefore, $B = \Sigma[\bigvee \omega]$. As a consequence, f is a weak bimorphism.

Let $\mathcal{L}_0 \in \mathbf{Cal^0_{Sym}}$ and let $g : \mathcal{L}_1 \times \mathcal{L}_2 \to \mathcal{L}_0$ be a weak bimorphism. Define $h : \mathcal{L}_1 \circledast \mathcal{L}_2 \to \mathcal{L}_0$ as $h(a) := \bigvee\{g(p)\,;\, p \in a\}$. For $p \in \Sigma_1 \times \Sigma_2$, define $g_{p_1} := g(p_1, -)$ and $g_{p_2} := g(-, p_2)$. Let $x \in \Sigma'_0$ the set of coatoms of $\mathcal{L}_0$. Denote by H, G_{p_1} and G_{p_2} the restrictions to atoms of h, g_{p_1} and g_{p_2} respectively. Then

$$H^{-1}(\Sigma[x] \bigcup \{0\}) = \bigcup_{p_1 \in \Sigma_1} p_1 \times G^{-1}_{p_1}(\Sigma[x] \bigcup \{0\}) = \bigcup_{p_2 \in \Sigma_2} G^{-1}_{p_2}(\Sigma[x] \bigcup \{0\}) \times p_2\,.$$

Now, since g is a weak bimorphism, $\bigvee G^{-1}_{p_i}(\Sigma[x] \bigcup \{0\})$ is a coatom or 1, therefore $H^{-1}(\Sigma[x] \bigcup \{0\}) \in \Sigma'_{\circledast} \bigcup \{1\}$. As a consequence, by Lemma 3.3 and Lemma 4.7, $h \in \mathbf{Cal^0_{Sym}}(\mathcal{L}_1 \circledast \mathcal{L}_2, \mathcal{L}_0)$. Let $h' \in \mathbf{Cal^0_{Sym}}(\mathcal{L}_1 \circledast \mathcal{L}_2, \mathcal{L}_0)$ such that $h' \circ f = g$. Then h' equals h on atoms, therefore $h' = h$. $\square$

Note that Lemma 4.9 may be proved directly by using this theorem.

Remark 7.4. In a category $\mathbf{C}$ concrete over $\mathbf{Set}$, define bimorphisms as maps $f : \mathsf{A} \times \mathsf{B} \to \mathsf{C}$ such that $f(a, -) \in \mathbf{C}(\mathsf{B}, \mathsf{C})$ and $f(-, b) \in \mathbf{C}(\mathsf{A}, \mathsf{C})$ for all $a \in \mathsf{A}$ and $b \in \mathsf{B}$. Moreover, define a tensor product as in Definition 7.1 with bimorphisms instead of weak bimorphisms. Then, for the category of join semilattices with maps preserving all finite joins, the definition of a tensor product is equivalent to the definition of the semilattice tensor product given by Fraser in [10] (note that $f(\mathcal{L}_1 \times \mathcal{L}_2)$ generates $\mathcal{L}$ if and only if the arrow h is unique). Note also that for the category of complete atomistic lattices with maps preserving arbitrary joins, the tensor product is given by $\circledcirc$ (the proof is similar to the proof of Theorem 7.3, see [12]).

8. The $\circledast$ and $\multimap$ products for DAC-lattices

Lemma 8.1. *Let $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$ be DAC-lattices, x_1, $y_1 \in \Sigma'_1$, and x_2, $y_2 \in \Sigma'_2$, and let $h : \Sigma'[x_1 \bigwedge y_1] \to \Sigma'[x_2 \bigwedge y_2]$ be a bijection. Then $x^h := \bigcup\{z \circ h(z)\,;\, z \in \Sigma'[x_1 \bigwedge y_1]\}$ is a coatom of $\mathcal{L}_1 \circledast \mathcal{L}_2$, which we call a $*-$coatom.*

Proof. Let $p \in \Sigma_1 \times \Sigma_2$. Since $\mathcal{L}_i$ and $\mathcal{L}_i^{\mathrm{op}}$ have the covering property, either $p_i \le x_i \bigwedge y_i$ or there is a unique $z_i \in \Sigma'[x_i \bigwedge y_i]$ such that $p_i \le z_i$. Therefore, either $x_2^h[p] = \Sigma_2$ or $x_2^h[p] = \Sigma[h(z_1)]$ for some $z_1 \in \Sigma'[x_1 \bigwedge y_1]$, and either $x_1^h[p] = \Sigma_1$ or $x_1^h[p] = \Sigma[h^{-1}(z_2)]$ for some $z_2 \in \Sigma'[x_2 \bigwedge y_2]$. As a consequence $x^h \in \Sigma'_{\circledast}$. □

Lemma 8.2. *Let $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$ be DAC-lattices and $f \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_1, \mathcal{L}_2^{\mathrm{op}})$ with $f(\mathcal{L}_1)$ of length 2. Let ξ be the bijection of Lemma 4.8. Then $\xi^{-1}(f)$ is a $*-$coatom.*

Proof. Since by hypothesis $f(\mathcal{L}_1)$ is of length 2, there exists x_2, $y_2 \in \Sigma'_2$ such that

$$f(\mathcal{L}_1) = [0_{\mathrm{op}}, x_2 {\textstyle\bigvee_{\mathrm{op}}} y_2]\,,$$

where $[0_{\mathrm{op}}, x_2 \bigvee_{\mathrm{op}} y_2]$ denotes the interval $\{a \in \mathcal{L}_2^{\mathrm{op}}\,;\, 0_{\mathrm{op}} \le_{\mathrm{op}} a \le_{\mathrm{op}} x_2 \bigvee_{\mathrm{op}} y_2\}$. We write $X := \xi^{-1}(f)$. Hence, by hypothesis, we have

$$1 \circ (x_2 {\textstyle\bigwedge} y_2) \subseteq X := \xi^{-1}(f)\,.$$

(1) **Claim**: $\forall z \in \Sigma'_2$, $1 \circ z \nsubseteq X$. [*Proof.* If $1 \circ z \subseteq X$ for some $z \in \Sigma'_2$, then $f(\mathcal{L}_1) = [0_{\mathrm{op}}, z]$, a contradiction, since by hypothesis, $f(\mathcal{L}_1)$ is of length 2.]

(2) **Claim**: $\forall x \in \Sigma'_1$, $x \circ 1 \nsubseteq X$. [*Proof.* Suppose that $x \circ 1 \subseteq X$ for some $x \in \Sigma'_1$. Let p be an atom of $\mathcal{L}_1$ not under x. Then X contains $(\Sigma[x] \bigcup p) \times \Sigma[f(p)]$, hence, since $\mathcal{L}_1 \circledast \mathcal{L}_2 \subseteq \mathcal{L}_1 \otimes \mathcal{L}_2$, it follows that $1 \circ f(p) \subseteq X$; whence a contradiction by part 1.]

(3) **Claim**: $\forall q \in \Sigma_2$, $f^\circ(q) \circ ((x_2 \bigwedge y_2) \bigvee q) \subseteq X$. [*Proof.* Let $q \in \Sigma_2$. For any atom p under $f^\circ(q)$, we have $q \le f(p)$, hence, since $1 \circ (x_2 \bigwedge y_2) \subseteq X$, $p \times (\Sigma[x_2 \bigwedge y_2] \bigcup q) \subseteq X$. As a consequence, $p \circ ((x_2 \bigwedge y_2) \bigvee q) \subseteq X$, since $\mathcal{L}_1 \circledast \mathcal{L}_2 \subseteq \mathcal{L}_1 \otimes \mathcal{L}_2$.]

(4) **Claim**: Let $q \in \Sigma_2$ with $q \bigwedge x_2 \bigwedge y_2 = 0$. Then, $f^\circ(q) \neq 1$. [*Proof.* Suppose that $f^\circ(q) = 1$. Then, by part 3, $1 \circ ((x_2 \bigwedge y_2) \bigvee q) \subseteq X$. Thus, $(x_2 \bigwedge y_2) \bigvee q \neq 1$. Moreover, since $\mathcal{L}_2^{\mathrm{op}}$ has the covering property, $(x_2 \bigwedge y_2) \bigvee q$ is a coatom; whence a contradiction by part 1.]

(5) Let z be a coatom of $\mathcal{L}_2$ above $x_2 \bigwedge y_2$. **Claim**: For any atoms p and q of $\mathcal{L}_2$ with p, $q \le z$ and $p \bigwedge x_2 \bigwedge y_2 = 0 = q \bigwedge x_2 \bigwedge y_2$, we have $f^\circ(p) = f^\circ(q)$. [*Proof.* Suppose that $f^\circ(p) \neq f^\circ(q)$. Now, $(x_2 \bigwedge y_2) \bigvee p = z = (x_2 \bigwedge y_2) \bigvee q$. Whence, by part 3, $(f^\circ(p) \bigvee f^\circ(q)) \circ z = 1 \circ z \subseteq X$, a contradiction by part 1.]

As a consequence, we can define a map $k : \Sigma'[x_2 \bigwedge y_2] \to \Sigma'_1$ as $k(z) := f^\circ(q)$ for any atom $q \le z$ such that $q \bigwedge x_2 \bigwedge y_2 = 0$. Note that $X = \{f^\circ(q) \circ q\,;\, q \in \Sigma_2\}$. Indeed, by Lemma 4.8, $X = \{p \circ f(p)\,;\, p \in \Sigma_1\}$. Now, $(r, s) \in X \Leftrightarrow s \in \Sigma[f(r)] \Leftrightarrow r \in \Sigma[f^\circ(s)]$. Hence, by what precedes, we have

$$X = 1 \circ (x_2 {\textstyle\bigwedge} y_2) \textstyle\bigcup \{k(z) \circ z\,;\, z \in \Sigma'[x_2 {\textstyle\bigwedge} y_2]\}\,.$$

(6) **Claim**: $\mathrm{Im}(k) \subseteq \Sigma'[k(x_2) \bigwedge k(y_2)]$. [*Proof.* First, note that

$$\Sigma[k(x_2) {\textstyle\bigwedge} k(y_2)] \times (\Sigma[x_2] \textstyle\bigcup \Sigma[y_2]) \subseteq X\,,$$

hence $k(x_2) \bigwedge k(y_2) \circ 1 \subseteq X$. Suppose that there is $z \in \Sigma'[x_2 \bigwedge y_2]$ such that $k(z) \ngeq k(x_2) \bigwedge k(y_2)$. Then $k(z) \bigwedge k(x_2) \neq k(z) \bigwedge k(y_2)$, and $k(z) \bigwedge k(x_2) \circ 1 \subseteq X$

and $k(z) \bigwedge k(y_2) \circ 1 \subseteq X$. Now, since $\mathcal{L}_1^{\mathrm{op}}$ has the covering property,

$$(k(z){\bigwedge}k(x_2)){\bigvee}(k(z){\bigwedge}k(y_2)) = k(z)\,,$$

therefore $k(z) \circ 1 \subseteq X$, a contradiction.]

(7) Let $p \in \Sigma_1$ with $p \bigwedge k(x_2) \bigwedge k(y_2) = 0$. **Claim**: $f(p) \neq 1$. [*Proof.* If $f(p) = 1$, then $(p \bigcup \Sigma[k(x_2) \bigwedge k(y_2)]) \times \Sigma_2 \subseteq X$, hence $(p \bigvee (k(x_2) \bigwedge k(y_2))) \circ 1 \subseteq X$. Now, $p \bigvee (k(x_2) \bigwedge k(y_2))$ is a coatom, whence a contradiction by part 2.]

(8) **Claim**: The map k is surjective. [*Proof.* Let z be a coatom of $\mathcal{L}_1$ above $k(x_2) \bigwedge k(y_2)$, and let p be an atom of $\mathcal{L}_1$ under z such that $p \bigwedge k(x_2) \bigwedge k(y_2) = 0$. By part 7, $f(p)$ is a coatom, and since $p \bigvee (k(x_2) \bigwedge k(y_2)) = z$, we find that $k(f(p)) = z$.]

(9) **Claim**: The map k is injective. [*Proof.* Let t and z be two coatoms above $x_2 \bigwedge y_2$. Suppose that $k(z) = k(t)$. Then $\Sigma[k(z)] \times (\Sigma[z] \bigcup \Sigma[t]) \subseteq X$, hence $k(z) \circ 1 \subseteq X$, a contradiction by part 2.] $\square$

Theorem 8.3 (Faure and Frölicher, [8] Theorem 10.1.3)**.** *For $i = 1$ and $i = 2$, let E_i be a vector space over a division ring $\mathbb{K}_i$, and $\mathsf{P}(E_i)$ the lattice of all subspaces of E_i. If $g : \mathsf{P}(E_1) \to \mathsf{P}(E_2)$ preserves arbitrary joins, sends atoms to atoms or 0, and if $g(\mathsf{P}(E_1))$ is of length ≥ 3, then g is induced by a semilinear map $f : E_1 \to E_2$ (i.e. $g(\mathbb{K}v) = \mathbb{K}f(v)$, $\forall v \in E_1$).*

Corollary 8.4. *For $i = 1$ and $i = 2$, let (E_i, F_i) be pairs of dual spaces, and let $g : \mathcal{L}_{F_1}(E_1) \to \mathcal{L}_{F_2}(E_2)$ be a join-preserving map, sending atoms to atoms or 0 with $g(\mathcal{L}_{F_1}(E_1))$ of length ≥ 3. Then there is a semilinear map $f : E_1 \to E_2$ that induces g.*

Proof. Define $h : \mathsf{P}(E_1) \to \mathsf{P}(E_2)$ as $h(V) = \bigvee g(\Sigma[V])$ where the join is taken in $\mathsf{P}(E_2)$. Note that on atoms $h = g$.

Denote by H the restriction of h to atoms and G the restriction of g to atoms (hence $G = H$). Let W be a subspace of E_2 and let $p, q \in H^{-1}(W)$. Then $h(p) \bigvee h(q)$ (where the join is taken in $\mathsf{P}(E_2)$) is a 2-dimensional subspace of E_2, hence $h(p) \bigvee h(q) \in \mathcal{L}_{F_2}(E_2)$ (see Remark 3.11). Therefore, by Lemma 3.3, $G^{-1}(h(p) \bigvee h(q)) \in \mathsf{Cl}(\mathcal{L}_{F_1}(E_1))$, hence $H^{-1}(h(p) \bigvee h(q)) \in \mathsf{Cl}(\mathsf{P}(E_1))$, therefore $\Sigma[p \bigvee q] \subseteq H^{-1}(h(p) \bigvee h(q))$. Moreover, from $h(p) \bigvee h(q) \subseteq W$ it follows that $H^{-1}(h(p) \bigvee h(q)) \subseteq H^{-1}(W)$. As a consequence,

$$\Sigma[p{\bigvee}q] \subseteq H^{-1}(h(p){\bigvee}h(q)) \subseteq H^{-1}(W)\,,$$

hence we have proved that $H^{-1}(W) \in \mathsf{P}(E_1)$. Therefore, it follows from Lemma 3.3 that h preserves arbitrary joins.

As a consequence, there exists a semilinear map $f : E_1 \to E_2$ that induces h, hence also g since h equals g on atoms. $\square$

Theorem 8.5. *Let $\mathcal{L}_1, \mathcal{L}_2 \in \mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$ be DAC-lattices of length ≥ 4, and X a coatom of $\mathcal{L}_1 \circledast \mathcal{L}_2$. Let (E_i, F_i) be pairs of dual spaces such that $\mathcal{L}_i \cong \mathcal{L}_{F_i}(E_i)$ (see Theorem 3.10). Let ξ be the bijection of Lemma 4.8. Then, X is a $*-$coatom, or there is a semilinear map g from E_1 to F_2 that induces $\xi(X)$. If $\xi(X)(\mathcal{L}_1)$ is of length 1, then X is a coatom of $\mathcal{L}_1 \circledS \mathcal{L}_2$.*

Proof. First note that $\mathcal{L}_2^{\mathrm{op}} \cong \mathcal{L}_{E_2}(F_2)$. Write $f = \xi(X)$. From Corollary 8.4, if $f(\mathcal{L}_1)$ is of length ≥ 3, there is a semilinear map g from E_1 to F_2 that induces f, whereas by Lemma 8.2, if $f(\mathcal{L}_1)$ is of length 2, X is a $*-$coatom.

Finally, if $f(\mathcal{L}_1) = [0_{\mathrm{op}}, x_2]$ for some coatom x_2 of $\mathcal{L}_2$, then $1 \circ x_2 \subseteq X$. Now, let $q \in \Sigma_2$ with $q \bigwedge x_2 = 0$. Then $f^\circ(q)$ is a coatom of $\mathcal{L}_1$. Moreover, $\Sigma[f^\circ(q)] \times (x_2 \bigcup q) \subseteq X$, hence $f^\circ(q) \circ 1 \subseteq X$. As a consequence, $f^\circ(q) \square x_2 \subseteq X$, hence by Lemma 4.6, $X = f^\circ(q) \square x_2$, therefore $X \in \Sigma'_{\owedge}$. $\square$

Theorem 8.6. *Let E_1 and E_2 be vector spaces of dimension n. Then there is an injective map from the set $\mathbb{P}(E_1 \otimes E_2)$ of one-dimensional subspaces of $E_1 \otimes E_2$ to the set of atoms of $\mathsf{P}(E_1) \multimap \mathsf{P}(E_2)$.*

Proof. Write $\mathcal{L}_1 := \mathsf{P}(E_1)$ and $\mathcal{L}_2 := \mathsf{P}(E_2)$. First, there is a bijection from $E_1 \otimes E_2$ to the set of linear maps between E_1 and E_2. Since both E_1 and E_2 are of dimension n, any linear map induces an arrow in $\mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_1, \mathcal{L}_2)$. As a consequence, there is an injective map from $\mathbb{P}(E_1 \otimes E_2)$ to $\mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_1, \mathcal{L}_2)$.

Now, by definition $\mathcal{L}_1 \multimap \mathcal{L}_2 = (\mathcal{L}_1 \circledast \mathcal{L}_2^{\mathrm{op}})^{\mathrm{op}}$, and by Lemma 4.8, there is a bijection between the set of coatoms of $\mathcal{L}_1 \circledast \mathcal{L}_2^{\mathrm{op}}$ and $\mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_1, \mathcal{L}_2) \backslash \{f_0\}$, where f_0 denotes the constant arrow which sends every atom of $\mathcal{L}_1$ to 0. As a consequence, there is a bijection between $\mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_1, \mathcal{L}_2) \backslash \{f_0\}$ and the set of atoms of $\mathcal{L}_1 \multimap \mathcal{L}_2$. $\square$

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Appendix A. The coherence conditions

The category $\langle \mathbf{C}, \otimes, \top, \alpha, l, r \rangle$ is monoidal if $l_\top = r_\top$, and for any objects A, B, C and D, the diagrams

$$\begin{tikzcd}
(\top \otimes \mathsf{A}) \otimes \mathsf{B} \arrow[rr, "\alpha_{\top\mathsf{AB}}"] \arrow[dr, "l_{\mathsf{A}} \otimes \mathrm{id}"'] & & \top \otimes (\mathsf{A} \otimes \mathsf{B}) \arrow[dl, "l_{\mathsf{A}\otimes\mathsf{B}}"] \\
& \mathsf{A} \otimes \mathsf{B} &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathsf{A} \otimes \top) \otimes \mathsf{B} \arrow[rr, "\alpha_{\mathsf{A}\top\mathsf{B}}"] \arrow[dr, "r_{\mathsf{A}} \otimes \mathrm{id}"'] & & \mathsf{A} \otimes (\top \otimes \mathsf{B}) \arrow[dl, "\mathrm{id} \otimes l_{\mathsf{B}}"] \\
& \mathsf{A} \otimes \mathsf{B} &
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathsf{A} \otimes \mathsf{B}) \otimes \top \arrow[rr, "\alpha_{\mathsf{AB}\top}"] \arrow[dr, "r_{\mathsf{A}\otimes\mathsf{B}}"'] & & \mathsf{A} \otimes (\mathsf{B} \otimes \top) \arrow[dl, "\mathrm{id} \otimes r_{\mathsf{B}}"] \\
& \mathsf{A} \otimes \mathsf{B} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
((\mathsf{A} \otimes \mathsf{B}) \otimes \mathsf{C}) \otimes \mathsf{D} \arrow[r, "\alpha_{\mathsf{A}\otimes\mathsf{BCD}}"] \arrow[d, "\alpha_{\mathsf{ABC}} \otimes \mathrm{id}"'] & (\mathsf{A} \otimes \mathsf{B}) \otimes (\mathsf{C} \otimes \mathsf{D}) \arrow[r, "\alpha_{\mathsf{ABC}\otimes\mathsf{D}}"] & \mathsf{A} \otimes (\mathsf{B} \otimes (\mathsf{C} \otimes \mathsf{D})) \\
(\mathsf{A} \otimes (\mathsf{B} \otimes \mathsf{C})) \otimes \mathsf{D} \arrow[rr, "\alpha_{\mathsf{AB}\otimes\mathsf{CD}}"'] & & \mathsf{A} \otimes ((\mathsf{B} \otimes \mathsf{C}) \otimes \mathsf{D}) \arrow[u, "\mathrm{id} \otimes \alpha_{\mathsf{BCD}}"']
\end{tikzcd}$$

commute (see [7], p. 472, or [13], §VII.1). Further, $\langle \mathbf{C}, \otimes, \top, \alpha, l, r, s \rangle$ is symmetric if the diagrams

$$\begin{tikzcd} \mathsf{A} \otimes \mathsf{B} \arrow[r, "s_{\mathsf{AB}}"] \arrow[dr, "\mathsf{id}"'] & \mathsf{B} \otimes \mathsf{A} \arrow[d, "s_{\mathsf{BA}}"] \\ & \mathsf{A} \otimes \mathsf{B} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{B} \otimes \top \arrow[rr, "s_{\mathsf{B}\top}"] \arrow[dr, "r_{\mathsf{B}}"'] & & \top \otimes \mathsf{B} \arrow[dl, "l_{\mathsf{B}}"] \\ & B & \end{tikzcd}$$

and

$$\begin{tikzcd} (\mathsf{A} \otimes \mathsf{B}) \otimes \mathsf{C} \arrow[r, "\alpha_{\mathsf{ABC}}"] \arrow[d, "s_{\mathsf{AB}} \otimes \mathsf{id}"'] & \mathsf{A} \otimes (\mathsf{B} \otimes \mathsf{C}) \arrow[r, "s_{\mathsf{AB}\otimes\mathsf{C}}"] & (\mathsf{B} \otimes \mathsf{C}) \otimes \mathsf{A} \arrow[d, "\alpha_{\mathsf{BCA}}"] \\ (\mathsf{B} \otimes \mathsf{A}) \otimes \mathsf{C} \arrow[r, "\alpha_{\mathsf{BAC}}"] & \mathsf{B} \otimes (\mathsf{A} \otimes \mathsf{C}) \arrow[r, "\mathsf{id} \otimes s_{\mathsf{AC}}"] & \mathsf{B} \otimes (\mathsf{C} \otimes \mathsf{A}) \end{tikzcd}$$

commute (see [13], §VII.7).

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Boris Ischi, Laboratoire de Physique des Solides, Université Paris-Sud, Bâtiment 510, 91405 Orsay, France

Current address: Collège de Candolle, 5 rue d'Italie, 1204 Geneva, Switzerland

E-mail address: boris.ischi@edu.ge.ch

Gavin J. Seal, School of Computer Science, McGill University, 3480 University Street, Montreal, QC H3A 2A7 Canada

E-mail address: gavin_seal@fastmail.fm